Leader Election in Rings

- leader el. impossible in anonymous rings
- asynch. alg with time compl. $O(n)$ and msg. compl. $O(n^2)$
- radius growth: time compl. $O(n)$, msg. compl. $O(n \cdot \log n)$

Lower Bound on Msg. Compl.

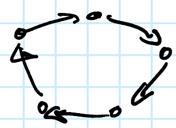
goal: Show that $\Omega(n \log n)$ msg. compl. is needed.

→ asynch. alg / uniform

→ max. ID node becomes leader } not really

→ everyone knows the ID of the leader } necessary

(→ directed ring (sense of dir.)

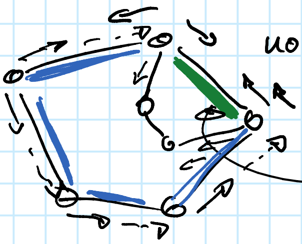

)

Open Schedule

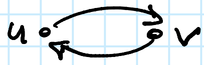
schedule: execution chosen the scheduler

schedule on a ring is open if there is an open edge

open edge: edge, where no message has been received so far



Ring with 2 nodes



$u < v$

ID of

→ v has to be the leader

→ u has to know v's ID

→ v has to send a msg. to u

stop execution as soon as the first msg. has been received

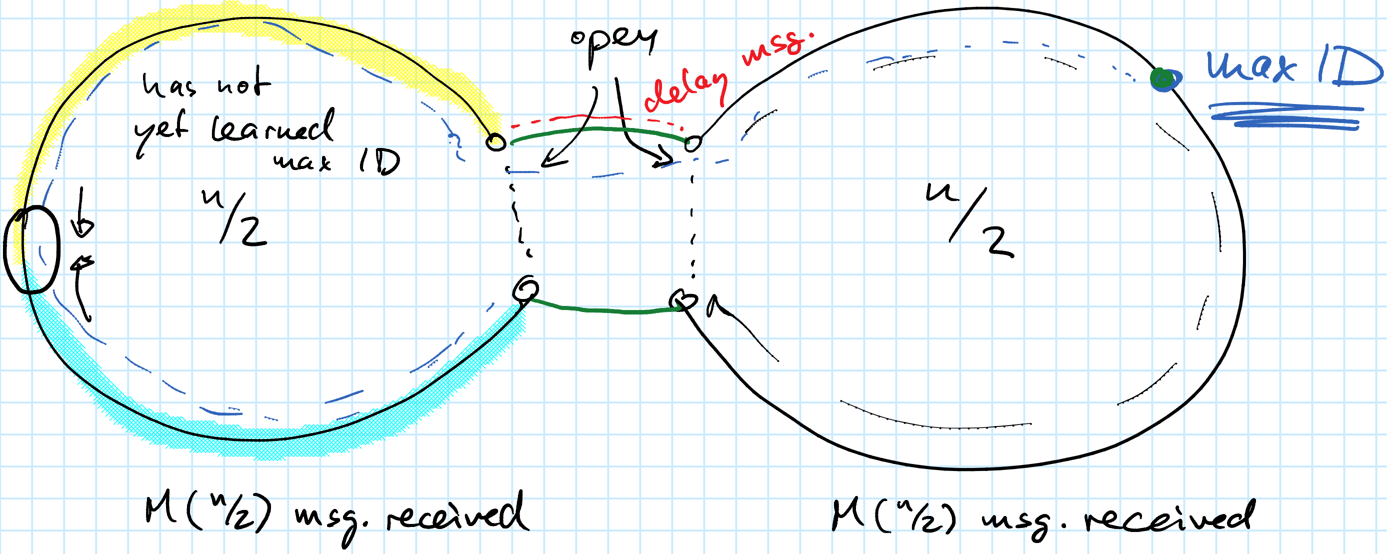
Lemma:
On a ring with 2 nodes, we can construct an open schedule in which at least one msg. has been received.

Goal: construct open schedule on ring R of size $n = 2^k$ on which as many msg. as possible have been rcv.

$M(n)$: # received msg. we can guarantee for open schedules on rings of size n

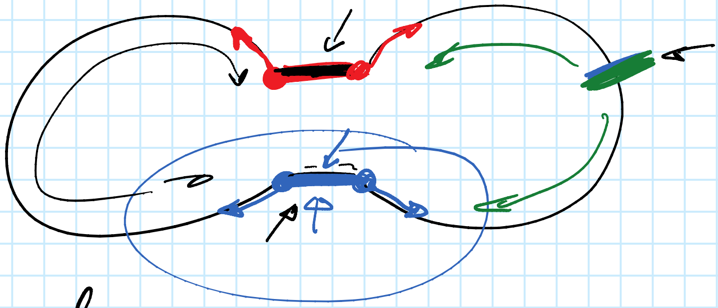
$(M(2) \stackrel{\checkmark}{=} 1)$

Lemma: $M(n) \geq 2M(n/2) + n/4$.
(for all $n = 2^k$, $k \geq 2$)



If we let the alg. run, at least $n/2$ more msg. will be received on the left side

delay msg. on red edge
if $< n/4$ are received



we can keep one of the two edges open and have $n/4$ msg. received. $\square$

$$M(n) \geq 2M(n/2) + n/4, \quad M(2) \geq 1$$

Thm: $M(n) \geq \frac{n}{4}(\log n + 1) = \Omega(n \log n)$

by induction

base: $M(2) \geq \frac{1}{2}(\underbrace{\log 2 + 1}_{2}) = 1$ ✓

step: $M(n) \geq 2\underbrace{M(n/2)}_{\text{I.H.}} + n/4$

$$\geq 2\left(\frac{n}{8}\left(\underbrace{\log \frac{n}{2} + 1}_{\log n}\right)\right) + \frac{n}{4}$$

$$= \frac{n}{4} \cdot \log n + \frac{n}{4} \quad ✓ \qquad \square$$

⟹ "every" uniform, asynchr. leader elec. alg. on the ring has msg. compl. $\Omega(n \log n)$

⟹ our radius growth alg. is asympt. optimal

What about synchr., non-unif. algorithms? solve leader election?

Only care about msg. compl.

assume all nodes start at time 0

IDs are natural numbers 1, 2, 3, ...

lowest ID node will become leader

Phases 1, 2, 3, ...

In phase i:

If ID = i then send "ID = i" around the ring } takes n rounds

Msg compl: n time compl. $O(n \cdot \min ID)$