Locality Lower Bounds

Thursday, July 10, 2014
9:02 AM

First lecture:

graph coloring : color rooted trees / oriented rings / rings / bounded degr. graphs / ...

$\Longrightarrow (\Delta+1)$-coloring in $O(\log^* n)$ time

rings: $\Delta+1 = 3$

general graphs: last week: $(\Delta+1)$-coloring in $O(\log n)$ rounds (randomized)

---

today's goal:

In the ring: $\log^* n$ rounds is optimal

\# times one needs to apply log to reach 1

Want to show: lower bound for 3 (c) - coloring in the ring

Assumptions

- deterministic algorithms
- synchronous algorithms
- msg. size / local comp. unbounded
- directed / oriented ring, n nodes
- unique IDs, IDs = $\{1, \ldots, n\}$

Need:

color with ≤ 3 (c) colors $\Longrightarrow > r(n)$ rounds

contraposition

$\leq r(n)$ rounds $\Longrightarrow$ need > 3 (c) colors

We need to understand

r-round algorithms

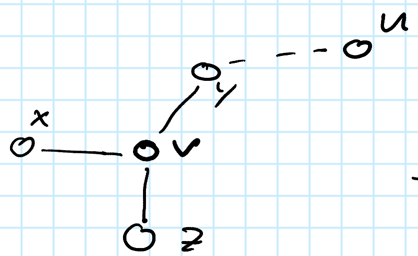
Idea: fix r

r-round alg need at least $f(r)$ colors

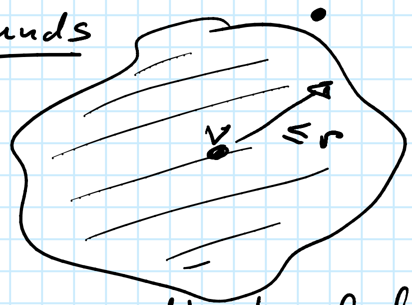
alg. assigns colors $\in \mathbb{N}$

$\Longrightarrow$ largest color the alg. can assign

r-round coloring algorithms



r rounds



r-hop neighborhood of v

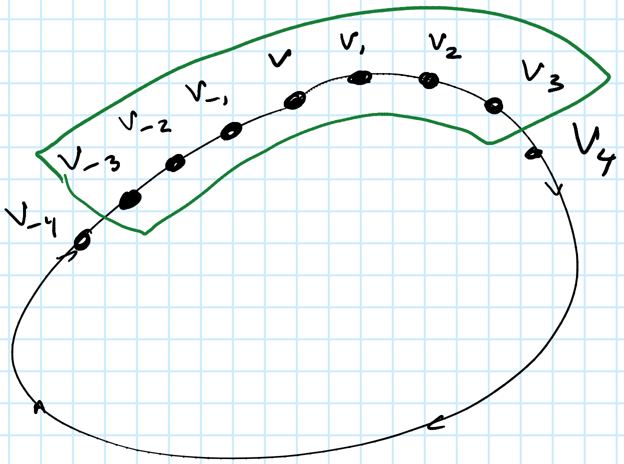
r-hop view of v

labeled r-hop neighborhood

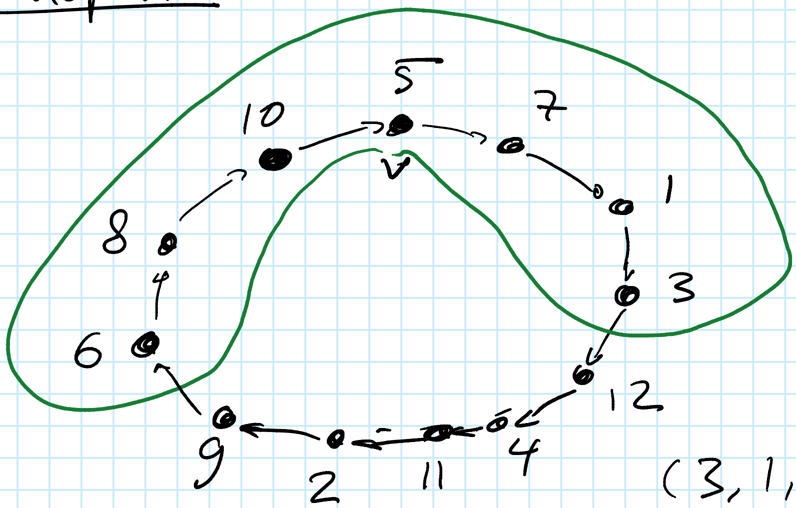
color of v is uniquely determined by the r-hop view of v

r-round coloring alg. function $\{\text{r-hop views}\} \longrightarrow \{1, \dots, c\}$

3 rounds



3-hop view



$(3, 1, 7, 5, 10, 8, 6)$

3-hop view of v : $\underline{(6, 8, 10, 5, 7, 1, 3)}$

r-hop view of v (ring of size n)

$(2r+1)$-tuple of IDs

$$\underline{(\ell_{-r}, \ldots, \ell_{-1}, \ell_0, \ell_1, \ldots, \ell_r)} \quad r\text{-hop views}$$

$$\ell_i \in \{1, \ldots, n\}, \quad \ell_i \neq \ell_j \text{ for } i \neq j$$

r round coloring alg. assigns a color to each possible such $(2r+1)$-tuple !

# of possible tuples: $\dfrac{n!}{(n-2r-1)!} \approx \underline{n^{2r+1}}$

In the example:

A $(6, 8, 10, 5, 7, 1, 3)$ needs to get a different color

than $(9, 6, 8, 10, 5, 7, 1)$ B

(but also : $(2, 6, 8, 10, 5, 7, 1)$ C

$(\ell_{-r}, \dots, \ell_0, \dots, \ell_r)$ and $(\ell'_{-r}, \dots, \ell'_0, \dots, \ell'_r)$
need different colors if there exists some ring where they appear next to each other

- $\underbrace{\begin{matrix}(\ell_{-r}, \ell_{-(r-1)}, \dots, \ell_0, \ell_1, \dots, \ell_r) \\ \neq (\ell'_{-r}, \dots, \ell'_{-1}, \ell'_0, \dots, \ell'_{r-1}, \ell'_r)\end{matrix}}$

  $\ell_i = \ell'_{i-1}$ for $i \in \{-(r-1), \dots, r\}$
  and $\underline{\ell_{-r} \neq \ell'_r}$

- symmetric case

Neighborhood Graph $\mathcal{N}_{r,n}$

$V[\mathcal{N}_{r,n}] := \{ (\ell_{-r}, \dots, \ell_0, \dots, \ell_r) : \ell_i \in \{1, \dots, n\}, \ell_i \neq \ell_j \}$

$E[\mathcal{N}_{r,n}] := \{$ pairs of consistent tuples $\}$

(r-round)
coloring algorithm is a proper coloring of $\mathcal{N}_{r,n}$

\# colors needed by an r-round coloring alg
$$\geq \chi(\mathcal{N}_{r,n})$$

$\Longrightarrow$ need a lower bound on $\chi(\mathcal{N}_{r,n})$!

degree of a node in $N_{r,n}$

$= 2 \cdot (n-2r-1)$

#edges of $N_{r,n} \approx n^{2r+2}$

$B_{k,n}$: directed graph

$V[B_{k,n}] = \{ \underbrace{(\alpha_1, \dots, \alpha_k)}_{} : \alpha_i \in \{1, \dots, n\}, \ \alpha_i < \alpha_{i+1} \}$

increasing k-tuples

edge from $(\alpha_1, \dots, \alpha_k)$ to $(\beta_1, \dots, \beta_k)$

iff for $i \in \{1, \dots, k-1\}$ $\beta_i = \alpha_{i+1}$

$(\alpha_1, \alpha_2, \dots, \alpha_k)$

$(\beta_1, \dots, \beta_{k-1}, \beta_k)$

$\Rightarrow$ incr. tuples $\Rightarrow \beta_k > \alpha_1$

Observation:

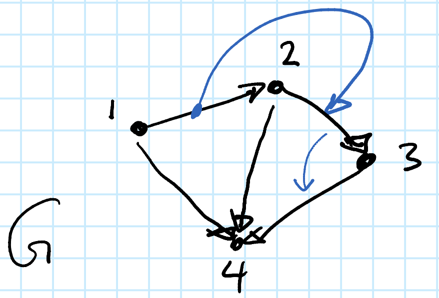
Viewed as an undir. graph, $B_{2r+1,n}$ is a subgraph of $N_{r,n}$

$$\Rightarrow \chi(N_{r,n}) \geq \underline{\underline{\chi(B_{2r+1,n})}}$$

Will get a lower bound of $\chi(B_{k,n})$ (for all k)

Diline Graph

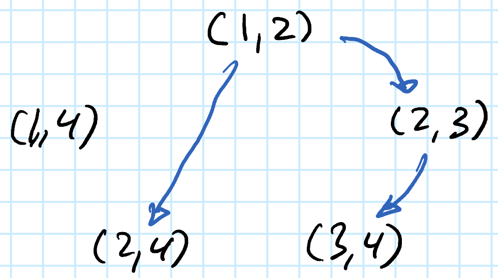
DL(G) of a dir. graph G is def as

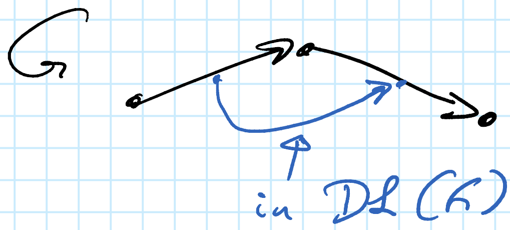


$V(G) = \{1, 2, 3, 4\}$

$E(G) = \{(1,2), (2,3), (1,4), (2,4), (3,4)\}$

DL(G) : $V(DL(G)) = E(G)$





Lets look at $DL(B_{k,n})$

$\vec{\alpha} = (\alpha_1, \dots, \alpha_k)$ edge iff $\beta_1 = \alpha_2, \beta_2 = \alpha_3, \dots$

$(\beta_1, \dots, \beta_{k-1}, \beta_k)$ edge iff $\gamma_1 = \beta_2, \gamma_2 = \beta_3, \dots$

$(\gamma_1, \dots \gamma_k)$

nodes $DL(B_{k,n}) = \{ (\underbrace{(\alpha_1, \dots, \alpha_k), (\beta_1, \dots, \beta_k)}_{(k+1)\text{-tuple}}), (\beta \dots), (\gamma \dots)), \dots \}$

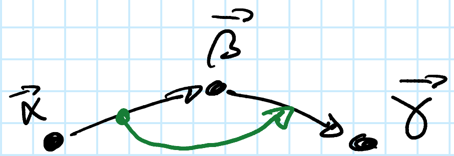
$(\alpha_1, \dots, \alpha_k, \beta_k)$

$V[DL(B_{k,n})]$

$= E[B_{k,n}] = \underbrace{\text{set of incr. } (k+1)\text{-tuples}}_{\text{nodes of } B_{k+1,n}}$

nodes of $DL(B_{k,n})$: $(k+1)$-tuples

edges of $DL(B_{k,n})$:



$\vec{\alpha} = (\alpha_1, \dots, \alpha_k)$

$\vdots$

nodes of $DL(B_{k,n}) = \{ (\vec{\alpha}, \vec{\beta}), \dots \}$

edges of $DL(B_{k,n}) = \{ \underbrace{(\vec{\alpha}, \vec{\beta}, \vec{\gamma})}_{(k+2)\text{-tuple!}}, \dots \}$

$$\Rightarrow \boxed{DL(B_{k,n}) \cong B_{k+1,n}}$$

Lemma: Let $G=(V,E)$ be a dir. graph

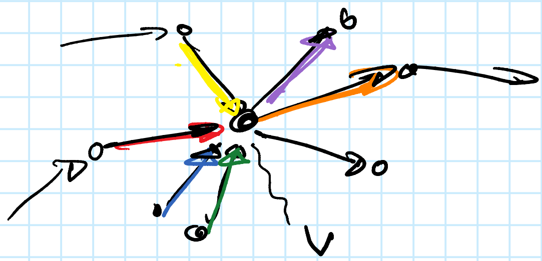
$$\chi(DL(G)) \geq \log_2(\chi(G))$$

Proof:

$$\chi(G) \leq 2^{\chi(DL(G))}$$

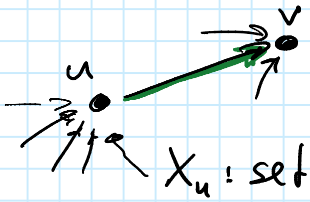
given a c-coloring of $\chi(DL(G))$

$\Rightarrow$ construct a 2^c-coloring of G



nodes u, v

X_v : set of incoming colors



claim: $X_u \neq X_v$

X_u : set of incoming colors

$\Longrightarrow$ X_u, X_v different

$\Longrightarrow$ use $\underline{\underline{X_u, X_v}}$ as colors for $\underline{\underline{u, v}}$

\# possible color sets: $\underline{\underline{2^c}}$

$\Downarrow$

$$\underline{\underline{\chi(B_{k+1,n})}} = \chi(DL(B_{k,n})) \geq \underline{\underline{\log_2(\chi(B_{k,n}))}}$$

<u>applied inductively</u>

apply log $(k-1)$ times

$$\chi(B_{k,n}) \geq \log_2^{(k-1)}(\underline{\underline{\chi(B_{1,n})}})$$

$\underline{\underline{B_{1,n}}}$ set of nodes: 1-tuples

$V(B_{1,n}) = \{1, \dots, n\}$

dir. edge from i to j iff $i<j$

$\Longrightarrow$ as undir. graph $B_{1,n} \cong K_n$

$\Longrightarrow$ $\chi(B_{1,n}) = n$

$\Longrightarrow$ $\chi(B_{k,n}) \geq \log_2^{(k-1)}(n)$

$\Longrightarrow$ $\chi(N_{r,n}) \geq \log_2^{(2r)}(n)$
$(k = 2r+1)$

<u>for which r is this > 3</u>

$\log^{(2r)}(n) > 3 \Longrightarrow 2r \leq \log^* n - 2$

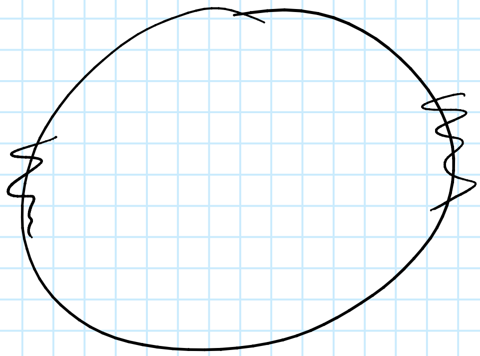
$\Longrightarrow r \leq \dfrac{\log^* n}{2} - 1$

$\#\text{rounds} \leq \frac{\log^* n}{2} - 1$

$\Longrightarrow\ > 3$ colors

3-coloring alg needs $\geq \frac{\log^* n}{2} - 1$ rounds

2-coloring? (if n is even)



$\chi(N_{r,n}) \geq 3$ if $r < \frac{n}{2} - \ldots$

(- - - - - -)

( )

— - .

(2, 7, 1, 5, 9)

(7, 1, 5, 9, 3)

(1, 5, 9, 3, 4)

(5, 9, 3, 4, 2)

...

same bound (asympt.) holds for randomized alg.

det. lower bound: Nati Linial

rand. " " : Moni Naor

$\Longrightarrow$ $\Omega(\log^* n)$ - lower bound on 3-coloring (in the ring) also gives an $\Omega(\log^* n)$ lower bound for computing an MIS in the ring.