Chapter 6
Consensus

Distributed Systems

SS 2015

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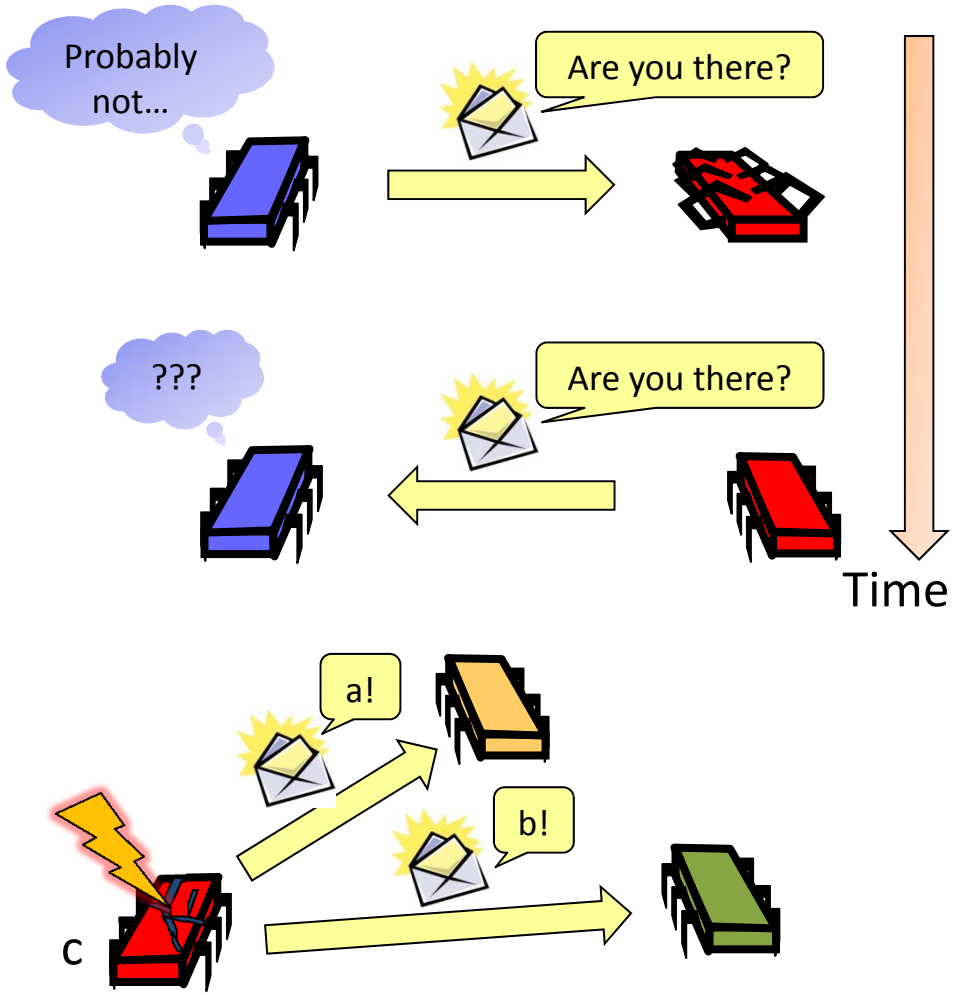
Overview

- Introduction
- Consensus #1: Shared Memory
- Consensus #2: Wait-free Shared Memory
- Consensus #3: Read-Modify-Write Shared Memory
- Consensus #4: Synchronous Systems
- Consensus #5: Byzantine Failures
- Consensus #6: A Simple Algorithm for Byzantine Agreement
- Consensus #7: The Queen Algorithm
- Consensus #8: The King Algorithm
- Consensus #9: Byzantine Agreement Using Authentication
- Consensus #10: A Randomized Algorithm
- Shared Coin

- Slides by R. Wattenhofer (ETHZ)

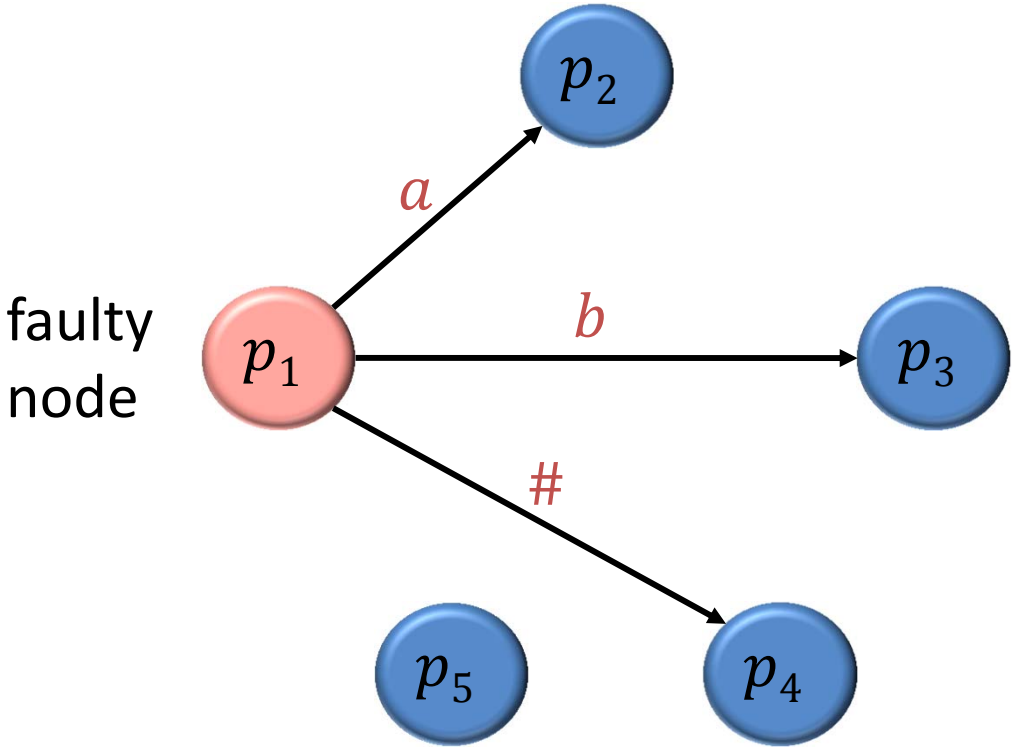
Arbitrary Behavior

- The assumption that processes crash and stop forever is sometimes too optimistic
- Maybe the processes fail and recover:
- Maybe the processes are damaged:

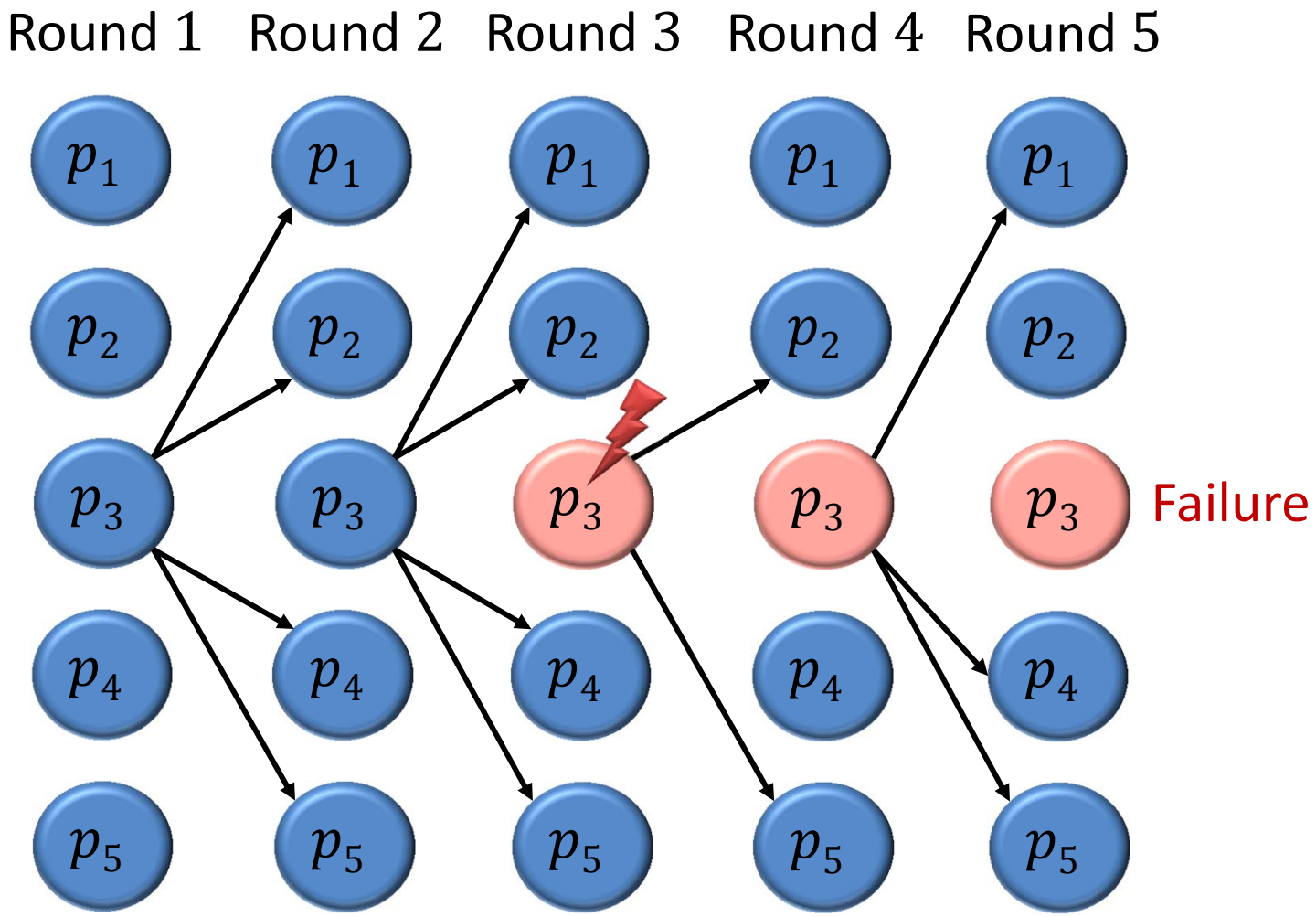


Consensus #5: Byzantine Failures

- Different processes may receive different values
- A Byzantine process can behave like a crash-failed process



After Failure, Node Remains in Network



Consensus with Byzantine Failures

- Again: If an algorithm solves consensus for f failed processes, we say it is an f-resilient consensus algorithm
- **Validity:** If all non-faulty processes start with the same value, then all non-faulty processes decide on that value
  - Note that in general this validity condition does not guarantee that the final value is an input value of a non-Byzantine process
  - However, if the input is binary, then the validity condition ensures that processes decide on a value that at least one non-Byzantine process had initially
- Obviously, any f-resilient consensus algorithm requires at least $f + 1$ rounds (follows from the crash failure lower bound)
- How large can f be…? Can we reach consensus as long as the majority of processes is correct (non-Byzantine)?

Impossibility

Theorem

There is no f-resilient Byzantine consensus algorithm for n nodes for $f \geq n/3$

Proof outline

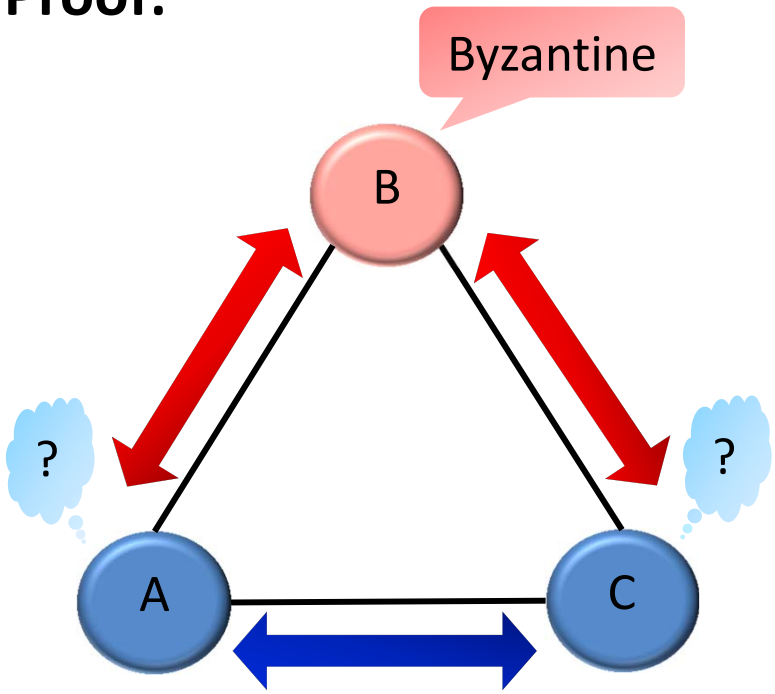
- First, we prove the 3 node case
  - not possible for $f = 1$
- The general case can then be proved by reduction from the 3 node case
  - Given an algorithm for n node and f faults for $f \geq n/3$, we can construct a 1-resilient 3-node algorithm

The 3 Node Case

Lemma

There is no 1-resilient algorithm for 3 nodes

Proof:

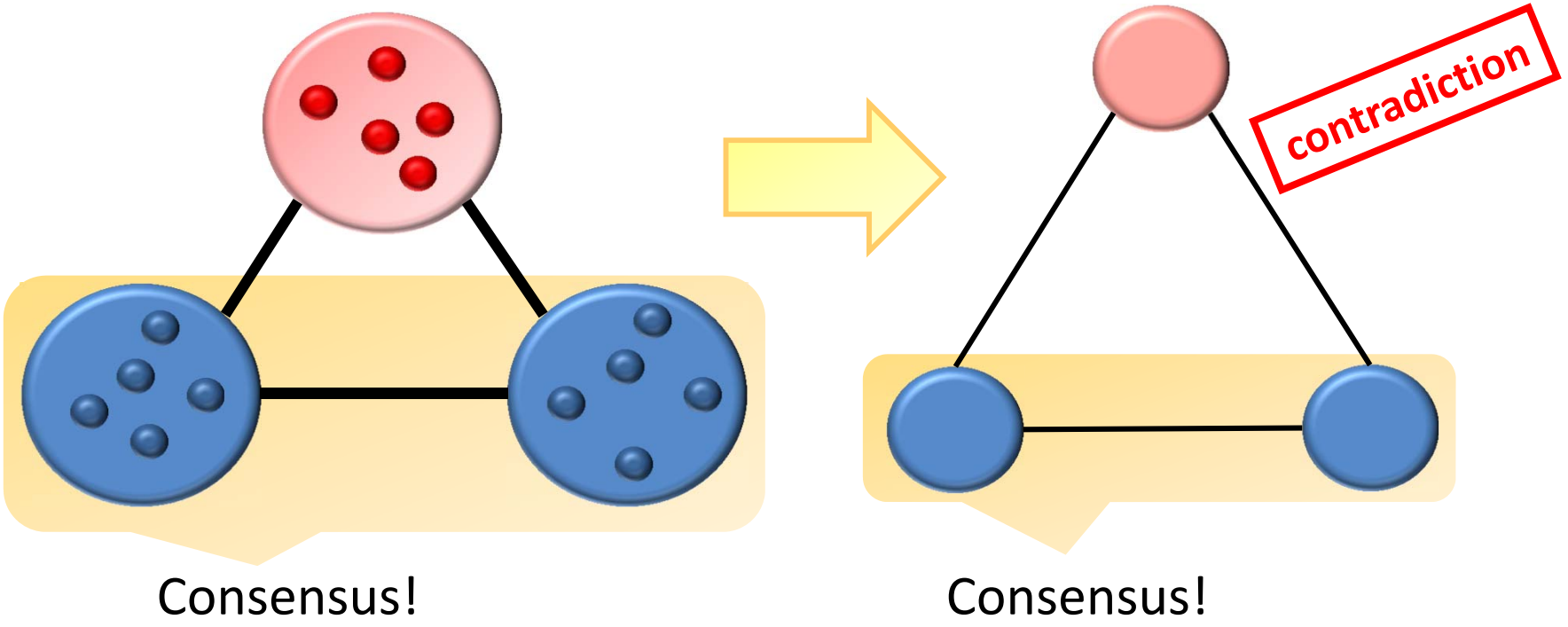


Intuition:

- Node A may also receive information from C about B's messages to C
- Node A may receive conflicting information about B from C and about C from B (the same for C!)
- It is impossible for A and C to decide which information to base their decision on!

The General Case

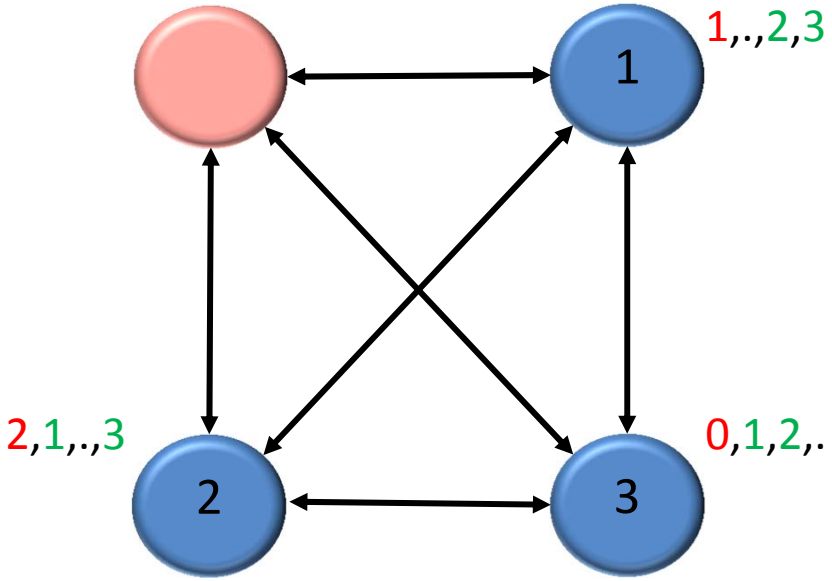
- One of the 3 nodes is Byzantine $\Longrightarrow$ its $n/3$ simulated nodes may all behave like Byzantine nodes
- Since algorithm A tolerates $n/3$ Byzantine failures, it can still reach consensus
  $\Longrightarrow$ We solved the consensus problem for three processes!



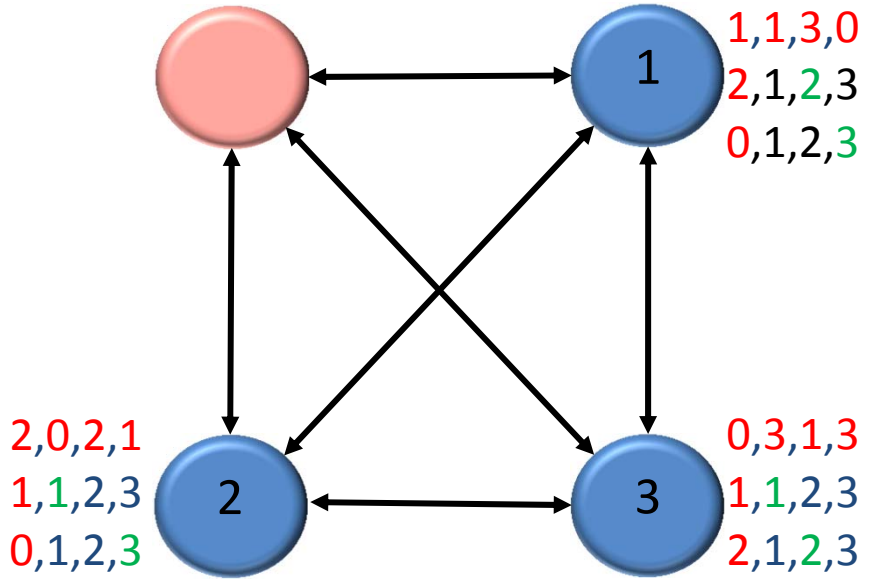
Cons. #6: Simple Byzantine Agreement Alg.

- Can the nodes reach consensus if $n > 3f$?
- A simpler question: What if $n = 4$ and $f = 1$?
- The answer is yes. It takes two rounds:

Round 1: Exchange all values



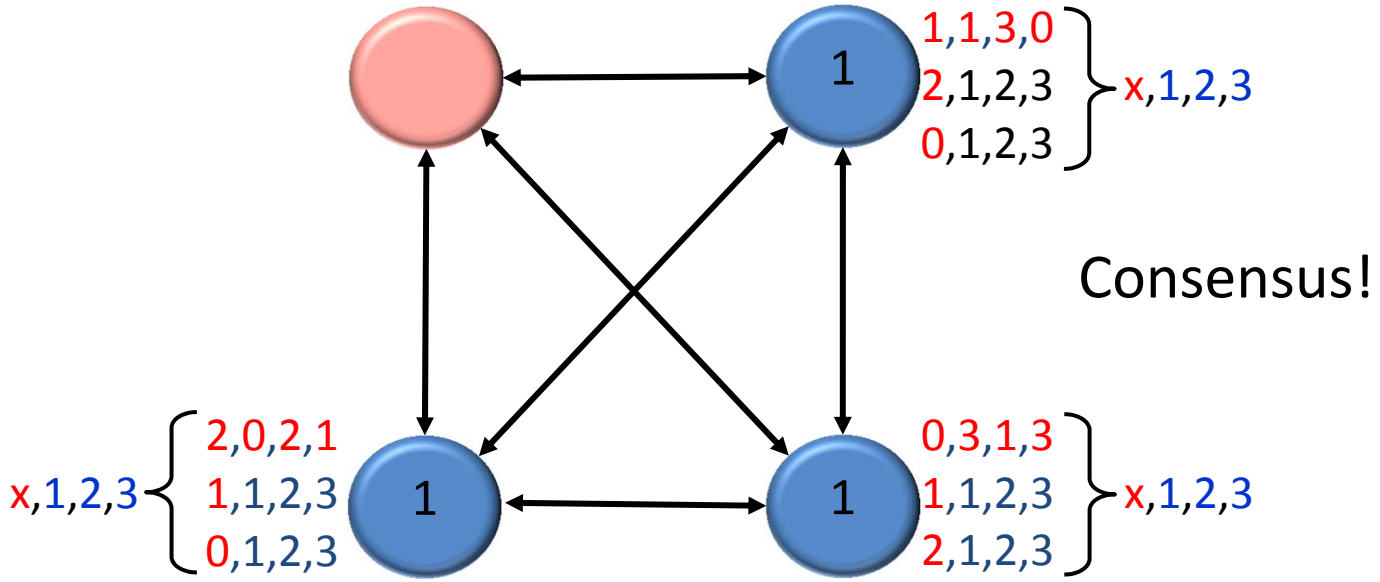
Round 2: Exchange received info



[matrix: one column for each original value, one row for each neighbor]

Simple Byzantine Agreement Algorithm

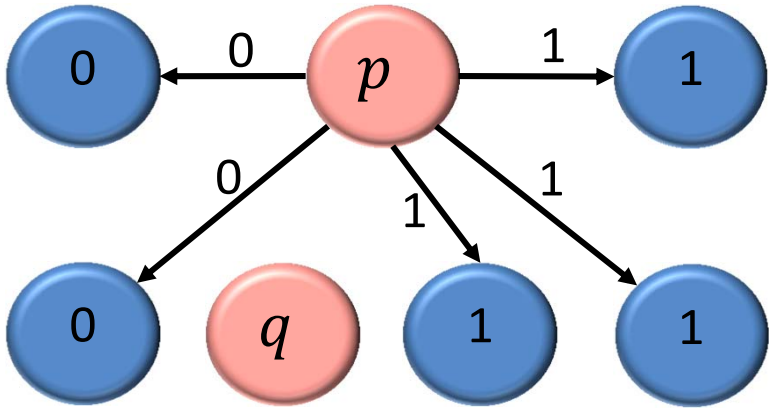
- After round 2, each node has received 12 values, 3 for each of the 4 input values (columns). If at least 2 of the 3 values of a column are equal, this value is accepted, otherwise it is discarded.
  - Values of honest nodes are accepted
  - The value of the Byzantine node is accepted iff it sends the same value to at least two nodes in the first round.
- Decide on most frequently accepted value!



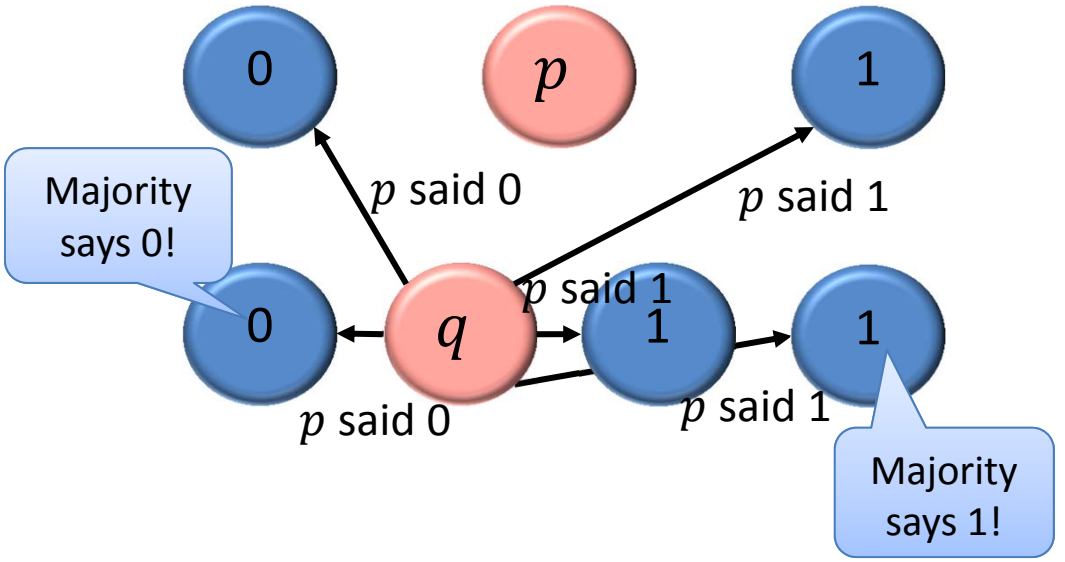
Simple Byzantine Agreement Algorithm

- Does the algorithm still work in general for any f and $n > 3f$?
- The answer is no. Try $f = 2$ and $n = 7$:

Round 1: Exchange all values



Round 2: Exchange received info



- The problem is that q can say different things about what p sent to q
  - What is the solution to this problem?

Simple Byzantine Agreement Algorithm

- The solution is simple: Again exchange all information!
- This way, the nodes can learn that q gave inconsistent information about p
- Hence, q can be excluded, and also p if it also gave inconsistent information (about q).
- If $f = 2$ and $n > 6$, consensus can be reached in 3 rounds!
- In fact, the following "algorithm" solves the problem for any f and any $n > 3f$:

Exchange all information for $f + 1$ rounds
Ignore all nodes that provided inconsistent information
Let all nodes decide based on the same input

Simple Byzantine Agreement Algorithm

The proposed algorithm has several advantages:

+ It works for any f and $n > 3f$, which is optimal
+ It only takes $f + 1$ rounds. This is even optimal for crash failures!
+ It works for any input and not just binary input

However, it has some considerable disadvantages:

– "Ignoring all nodes that provided inconsistent information'' is not easy to formalize

– The **size of the messages increases exponentially**!
This is a severe problem. It is therefore worth studying whether it is possible to solve the problem with small(er) messages

Consensus #7: The Queen Algorithm

- The Queen algorithm is a simple Byzantine agreement algorithm that uses small messages
- The Queen algorithm solves consensus with n nodes and f failures where $f < n/4$ in $f + 1$ phases

A phase consists of 2 rounds

Idea:

- There is a different (a priori known) queen in each phase
- Since there are $f + 1$ phases, in one phase the queen is not Byzantine
- Make sure that in this round all nodes choose the same value and that in future rounds the nodes do not change their values anymore

The Queen Algorithm

In each phase $i \in \{1, \dots, f+1\}$:

At the end of phase $f+1$, decide on own value

Round 1:

Broadcast own value

Also send own value to oneself

Set own value to the value that was received most often

If own value appears $> n/2 + f$ times

    support this value

else

    do not support any value

If several values have the same (highest) frequency, choose any value, e.g., the smallest

Round 2:

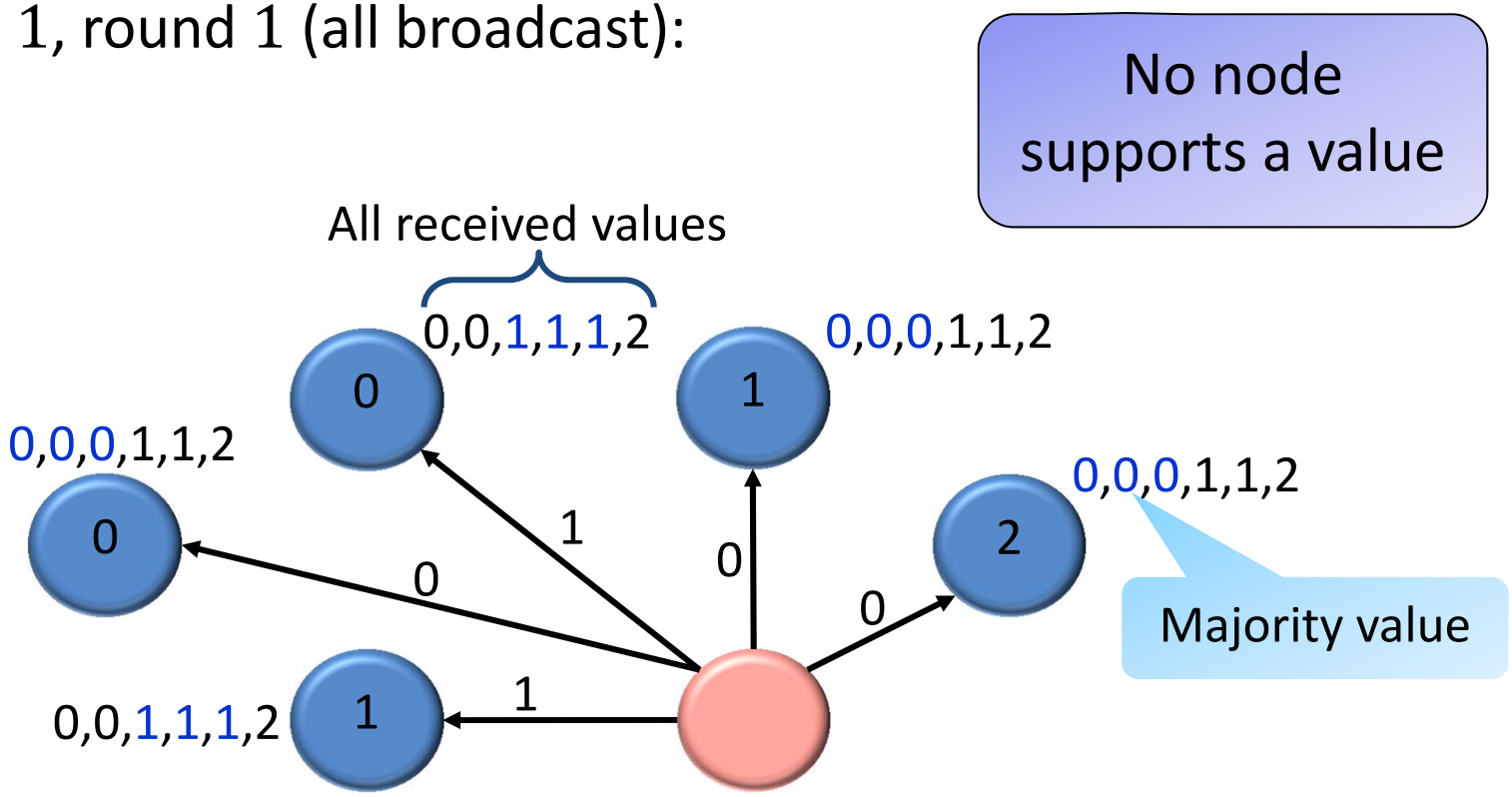
The queen broadcasts its value

If not supporting any value

    set own value to the queen's value

The Queen Algorithm: Example

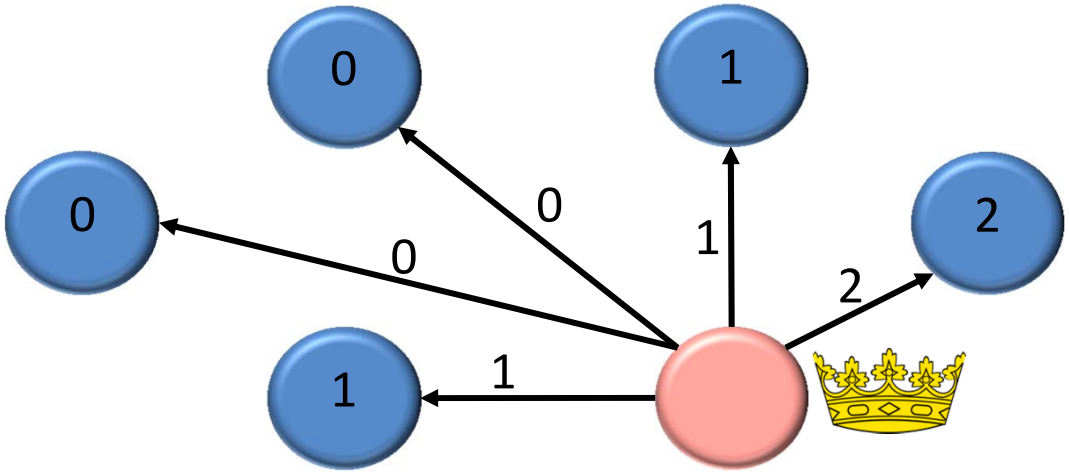
- Example: $n = 6, f = 1$
- Phase 1, round 1 (all broadcast):



The Queen Algorithm: Example

- Example: $n = 6, f = 1$
- Phase 1, round 2 (queen broadcasts):

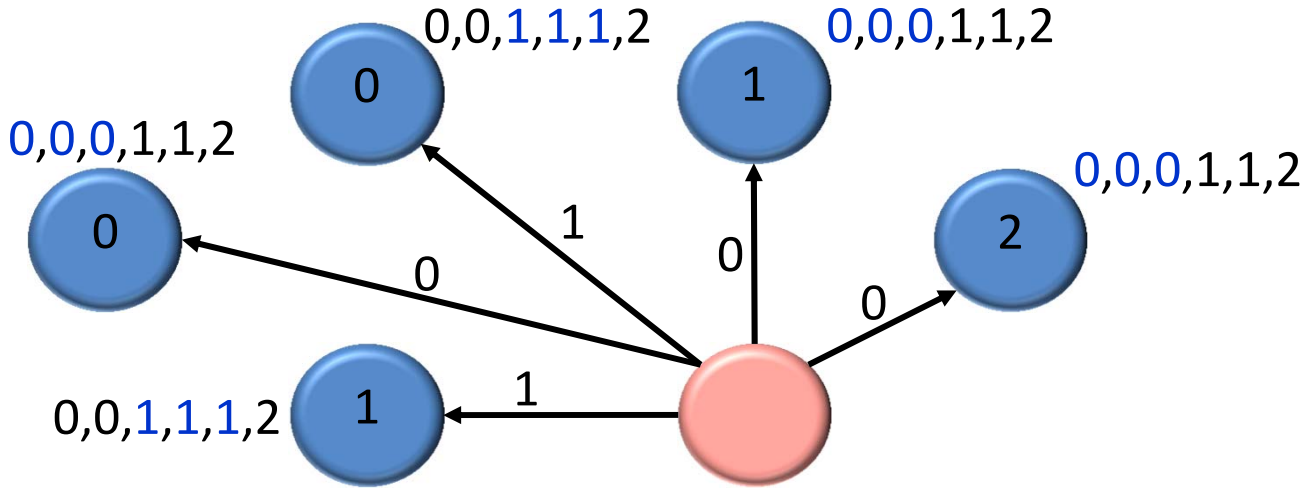
All nodes choose the queen's value



The Queen Algorithm: Example

- Example: $n = 6, f = 1$
- Phase 2, round 1 (all broadcast):

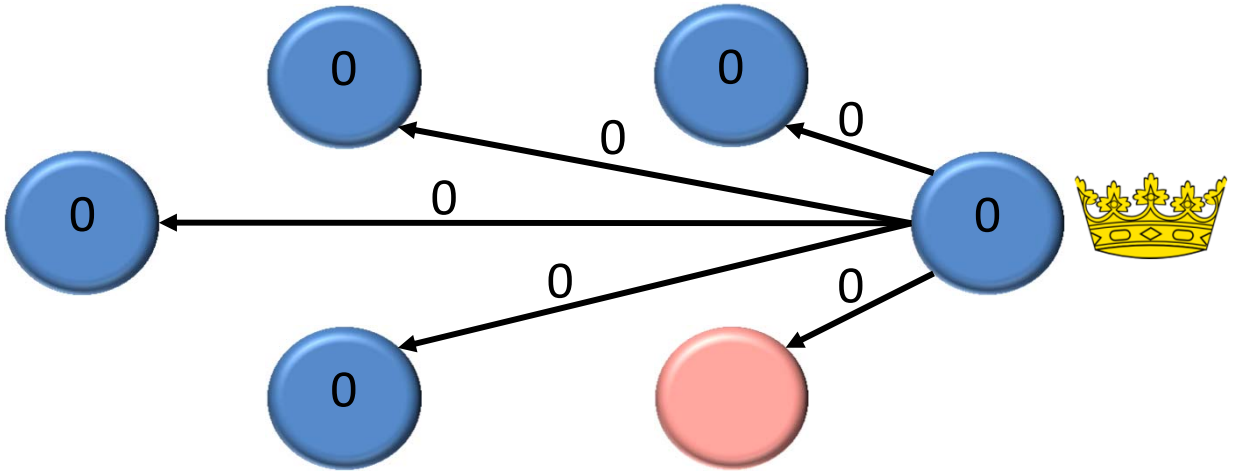
No node supports a value



The Queen Algorithm: Example

- Example: $n = 6, f = 1$
- Phase 2, round 2 (queen broadcasts):

All nodes choose the queen's value



Consensus!

The Queen Algorithm: Analysis

- After the phase where the queen is correct, all correct nodes have the same value
  - If all nodes change their values to the queen's value, obviously all values are the same
  - If some node does not change its value to the queen's value, it received a value $> n/2 + f$ times → All other correct nodes (including the queen) received this value $> n/2$ times and thus all correct nodes share this value
- In all future phases, no node changes its value
  - In the first round of such a phase, nodes receive their own value from at least $n - f > n/2$ nodes and thus do not change it
  - The nodes do not accept the queen's proposal if it differs from their own value in the second round because the nodes received their own value at least $n - f > n/2 + f$ times. Thus, all correct nodes support the same value

That's why we need $f < n/4$!

The Queen Algorithm: Summary

The Queen algorithm has several advantages:

+ The messages are small: nodes only exchange their current values
+ It works for any input and not just binary input

However, it also has some disadvantages:

– The algorithm requires $f+1$ phases consisting of 2 rounds each ... this is twice as much as an optimal algorithm
– It only works with $f < n/4$ Byzantine nodes!

- Is it possible to get an algorithm that works with $f < n/3$ Byzantine nodes and uses small messages?

Consensus #8: The King Algorithm

- The King algorithm is an algorithm that tolerates $f < n/3$ Byzantine failures and uses small messages
- The King algorithm also takes $f + 1$ phases

A phase now consists of 3 rounds

Idea:

- The basic idea is the same as in the Queen algorithm
- There is a different (a priori known) king in each phase
- Since there are $f + 1$ phases, in one phase the king is not Byzantine
- The difference to the Queen algorithm is that the correct nodes only propose a value if many nodes have this value, and a value is only accepted if many nodes propose this value

The King Algorithm

In each phase $i \in \{1 \dots f+1\}$:

At the end of phase $f+1$, decide on own value

Round 1:

Broadcast own value

Also send own value to oneself

Round 2:

If some value x appears $\geq n - f$ times

Broadcast "Propose x"

If some proposal received $> f$ times

Set own value to this proposal

Round 3:

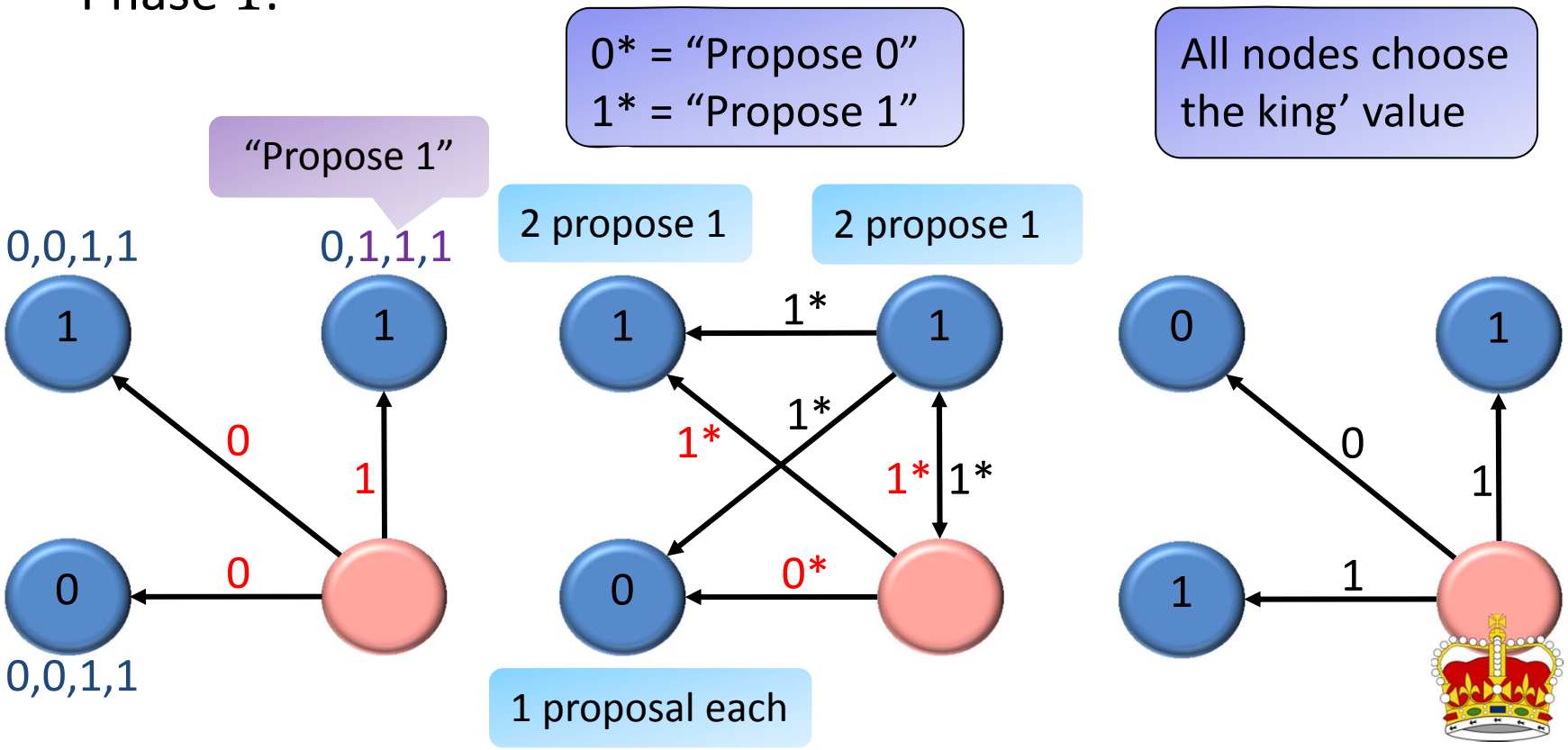
The king broadcasts its value

If own value received $< n - f$ proposals

Set own value to the king's value

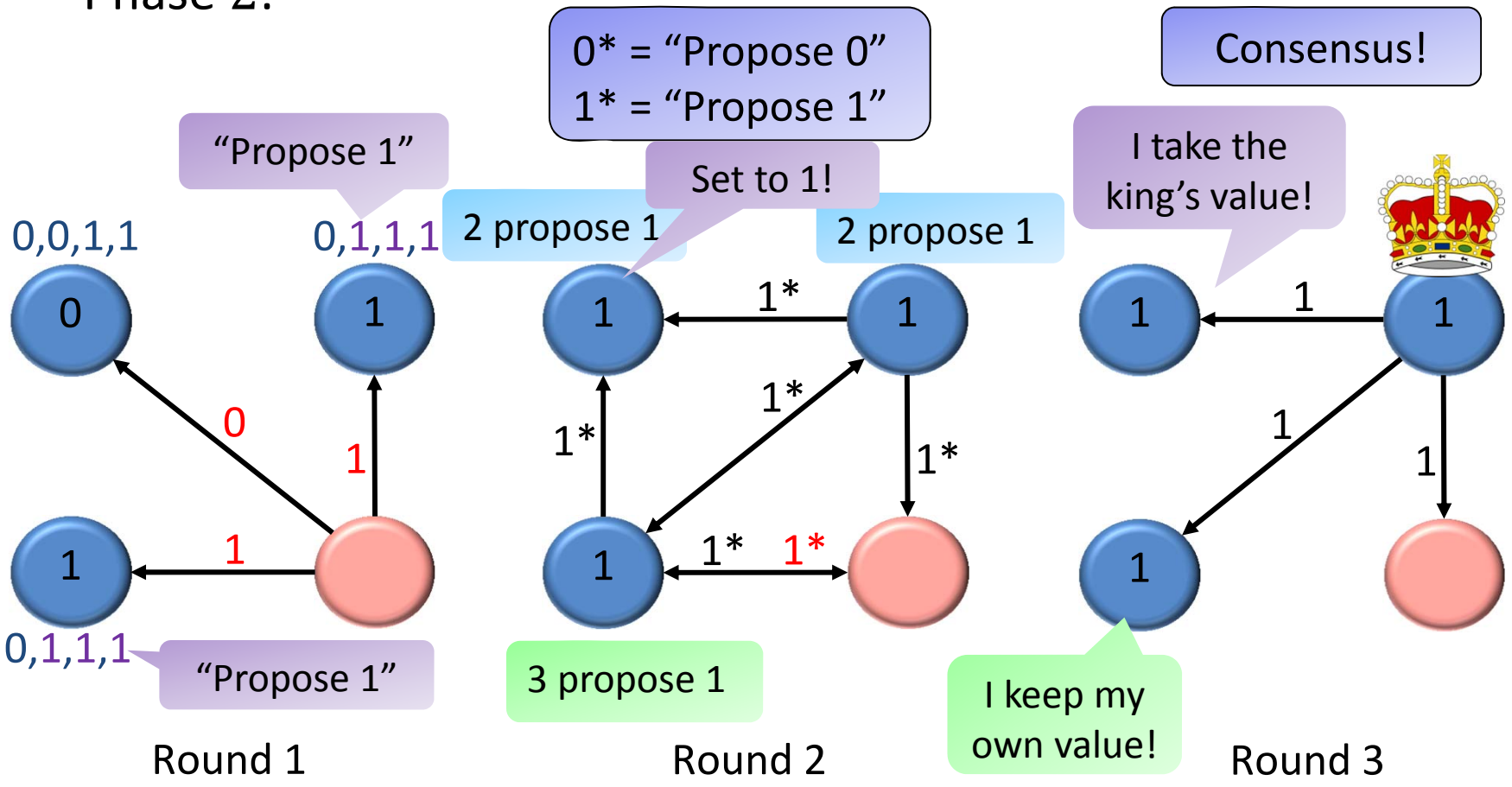
The King Algorithm: Example

- Example: $n = 4, f = 1$
- Phase 1:



The King Algorithm: Example

- Example: $n = 4, f = 1$
- Phase 2:



The King Algorithm: Analysis

- Observation: If some correct node proposes x, then no other correct node proposes $y \neq x$
  - Both nodes would have to receive $\geq n - f$ times the same value, i.e., both nodes received their value from $\geq n - 2f$ distinct correct nodes
  - In total, there must be $\geq 2(n - 2f) + f > n$ nodes, a contradiction!

We used that $f < n/3$!

- The validity condition is satisfied
  - If all correct nodes start with the same value, all correct nodes receive this value $\geq n - f$ times and propose it
  - All correct nodes receive $\geq n - f$ times proposals, i.e., no correct node will ever change its value to the king's value

The King Algorithm: Analysis

- After the phase where the king is correct, all correct processes have the same value
  - If all processes change their values to the king's value, obviously all values are the same
  - If some process does not change its value to the king's value, it received a proposal $\geq n - f$ times → $\geq n - 2f$ correct processes broadcast this proposal and all correct processes receive it $\geq n - 2f > f$ times
    → All correct processes set their value to the proposed value. Note that only one value can be proposed $> f$ times, which follows from the observation on the previous slide
- In all future phases, no process changes its value
  - This follows immediately from the fact that all correct processes have the same value after the phase where the king is correct and the validity condition

The King Algorithm: Summary

The King algorithm has several advantages:

+ It works for any f and $n > 3f$, which is optimal
+ The messages are small: processes only exchange their current values
+ It works for any input and not just binary input

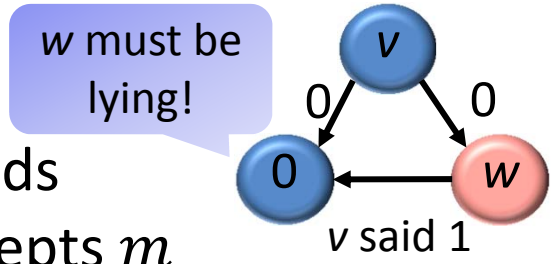
However, it also has a disadvantage:

– The algorithm requires $f + 1$ phases consisting of 3 rounds each
This is three times as much as an optimal algorithm

- Is it possible to get an algorithm that uses small messages and requires fewer rounds of communication?

Consensus #9: B. A. Using Authentication

- A simple way to reach consensus is to use authenticated messages



- Unforgeability condition: If a node never sends a message m, then no correct node ever accepts m

- Why is this condition helpful?
  - A Byzantine node cannot convince a correct node that some other correct node voted for a certain value if they did not!

Idea:

- There is a designated node v. The goal is to decide on v's value
- For the sake of simplicity, we assume a binary input. The default value is 0: if v cannot convince the nodes that v's input is 1, everybody chooses 0

Byzantine Agreement Using Authentication

If I am v and own input is 1
    value $:= 1$
    broadcast "v has 1"
else
    value $:= 0$

In each round $r \in \{1, \dots, r+1\}$:
If value $= 0$ and accepted r messages "v has 1" in total including a message from v itself
    value $:= 1$
    broadcast "v has 1" plus the r accepted messages that caused the local value to be set to 1

After $r + 1$ rounds:
Decide value

In total $r + 1$ authenticated "v has 1" messages

Byz. Agr. Using Authentication: Analysis

- **Assume that v is correct**
  - v's input is 1: All correct nodes accept v's message in round 1 and set value to 1. No node ever changes its value back to 0
  - v's input is 0: v never sends a message "v has 1", thus no correct process ever sets its value to 1
- **Assume that v is Byzantine**
  - v tries to convince some correct nodes that its input is 1
  - Assume that a correct node u sets its value to 1 in a round $r < f + 1$: Node u has accepted r messages including the message from v. Therefore, all other correct nodes accept the same r messages plus u's message and set their values to 1 as well in round $r + 1$
  - Assume that a correct node u sets its value to 1 in round $f + 1$: In this case, u accepted $f + 1$ messages. At least one of those is sent by a correct node, which must have set its value to 1 in an earlier round. We are again in the previous case, i.e., all correct nodes decide 1!

Byz. Agr. Using Authentication: Summary

Using authenticated messages has several advantages:

+ It works for any number of Byzantine nodes!
+ It only takes $f + 1$ rounds, which is optimal
+ Small messages: nodes send at most $f + 1$ "short" messages to all other nodes in a single round

sub-exponential length

However, it also has some disadvantages:

– If v is Byzantine, the nodes may agree on a value that is not in the original input
– It only works for binary input
– The algorithm requires authenticated messages…

B. A. Using Authentication: Improvements

- Can we modify the alg. so that it satisfies the validity condition?
  - Yes! Run the algorithm in parallel for $2f + 1$ "masters" v. Either 0 or 1 occurs at least $f + 1$ times, i.e., at least one correct node had this value. Decide on this value!
  - Alas, this modified protocol only works if $f < n/2$
- Can we modify the algorithm so that it also works with an arbitrary input?
  - Yes! In fact, the algorithm does not have to be changed much
  - We won't discuss this modification in class
- Can we get rid of the authentication?
  - Yes! Use *consistent broadcast*. This technique is not discussed either
  - This modified protocol works if $f < n/3$, which is optimal
  - However, each round is split into two
    → The total number of rounds is $2f + 2$

Consensus #10: A Randomized Algorithm

- So far we mainly tried to reach consensus in synchronous systems. The reason is that no deterministic algorithm can guarantee consensus in asynchronous systems even if only one process may crash

Synchronous system: Communication proceeds in synchronous rounds

- Can one solve consensus in asynchronous systems if we allow our algorithms to use randomization?

Asynchronous system: Messages are delayed indefinitely

- The answer is yes!
- The basic idea of the algorithm is to push the initial value. If other nodes do not follow, try to push one of the suggested values randomly
- For the sake of simplicity, we assume that the input is binary and at most $f < n/9$ nodes are Byzantine

Randomized Algorithm

$x :=$ own input; $r := 0$
Broadcast proposal(x, r)

In each round $r = 1, 2, \ldots$:

Wait for $n - f$ proposals
If at least $n - 2f$ proposals have some value y
    $x := y$; decide on y
else if at least $n - 4f$ proposals have some value y
    $x := y$;
else
    choose x randomly with $\Pr[x = 0] = \Pr[x = 1] = 1/2$
Broadcast proposal(x, r)
If decided on a value → stop

Randomized Algorithm: Analysis

Validity condition (If all have the same input, all choose this value)

- If all correct nodes have the same initial value x, they will receive $n - 2f$ proposals containing x in the first round and they will decide on x

Agreement (if the nodes decide, they agree on the same value)

- Assume that some correct node decides on x. This node must have received x from $n - 3f$ correct nodes. Every other correct node must have received x at least $n - 4f$ times, i.e., all correct nodes set their local value to x, and propose and decide on x in the next round

The processes broadcast at the end of a phase to ensure that the processes that have already decided broadcast their value again!

Randomized Algorithm: Analysis

Termination (all correct processes eventually decide)

- If some nodes do not set their local value randomly, they set their local value to the same value.
  Proof: Assume that some nodes set their value to 0 and some others to 1, i.e., there are $\geq n - 5f$ correct nodes proposing 0 and $\geq n - 5f$ correct processes proposing 1.
  Then, in total there are $\geq 2(n - 5f) + f > n$ nodes. Contradiction!

  That's why we need $f < n/9$!

  - Thus, in the worst case all $n - f$ correct nodes need to choose the same bit randomly, which happens with probability $1/2^{n-f}$
  - Hence, all correct processes eventually decide. The expected running time is smaller than 2^n
- The running time is awfully slow. Is there a clever way to speed up the algorithm?
- What about simply setting $x := 1$?! (Why doesn't it work?)

Can we do this faster?! Yes, with a Shared Coin

- A better idea is to replace

  choose x randomly with $\Pr[x = 0] = \Pr[x = 1] = 1/2$

  with a subroutine in which all the processes compute a so-called shared (a.k.a. common, "global") coin

- A shared coin is a random binary variable that is 0 with constant probability and 1 with constant probability
- For the sake of simplicity, we assume that there are at most $f < n/3$ crash failures (no Byzantine failures!!!)

All correct nodes know the outcome of the shared coin toss after each execution of the subroutine

Shared Coin Algorithm

Code for process i:

Set local coin $c_i := 0$ with probability $1/n$, else $c_i := 1$
Broadcast c_i
Wait for exactly $n - f$ coins and collect all coins in the local coin set s_i
Broadcast s_i
Wait for exactly $n - f$ coin sets
If at least one coin is 0 among all coins in the coin sets
    return 0
else
    return 1

Assume the worst case:
Choose f so that $3f + 1 = n$!

Shared Coin: Analysis

- Termination (of the subroutine)
  - All correct nodes broadcast their coins. It follows that all correct nodes receive at least $n - f$ coins
  - All correct processes broadcast their coin sets. It follows that all correct processes receive at least $n - f$ coin sets and the subroutine terminates
- We will now show that at least 1/3 of all coins are seen by everybody

A coin is *seen* if it is in at least one received coin set

- More precisely: We will show that at least $f + 1$ coins are in at least $f + 1$ coin sets
  - Recall that $3f + 1 = n$ and therefore $f + 1 > n/3$
  - Since these coins are in at least $f + 1$ coin sets and all processes receive $n - f$ coin sets, all correct processes see these coins!

Shared Coin: Analysis

- Proof that at least $f+1$ coins are in at least $f+1$ coin sets
  - Draw the coin sets and the contained coins as a matrix
  - Example: $n = 7, f = 2$

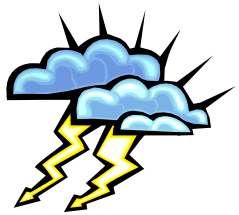
x means coin c_i is in set s_j

	s_1	s_3	s_5	s_6	s_7
c_1	x	x	x	x	x
c_2		x	x		
c_3	x	x	x	x	x
c_4		x	x		x
c_5	x			x	
c_6	x		x	x	x
c_7	x	x		x	x

Shared Coin: Analysis

- At least $f+1$ rows (coins) have at least $f+1$ x's (are in at least $f+1$ coin sets)
  - First, there are exactly $(n-f)^2$ x's in this matrix
  - Assume that the statement is wrong: Then at most f rows may be full and contain $n-f$ x's. And all other rows (at most $n-f$) have at most f x's
  - Thus, in total we have at most $f(n-f)+(n-f)f = 2f(n-f)$ x's
  - But $2f(n-f) < (n-f)^2$ because $2f < n-f$

Here we use $3f < n$



	s_1	s_3	s_5	s_6	s_7
c_1	x	x	x	x	x
c_2		x	x		
c_3	x	x	x	x	x
c_4		x	x		x
c_5	x			x	
c_6	x		x	x	x
c_7	x	x		x	x

Shared Coin: Theorem

Theorem

All processes decide 0 with constant probability, and all processes decide 1 with constant probability.

Proof:

- With probability $(1-1/n)^n \approx 1/e \approx 0.37$ all nodes choose 1. Thus, all correct nodes return 1
- There are at least $f+1 \approx n/3$ coins seen by all correct nodes. The probability that at least one of these coins is set to 0 is at least

$$1-\left(1-\frac{1}{n}\right)^{\frac{n}{3}} \approx 1-\left(1-\frac{1}{e}\right)^{\frac{1}{3}} \approx 0.28$$

Back to Randomized Consensus

- If this shared coin subroutine is used, there is a constant probability that the processes agree on a value
- Some nodes may not want to perform the subroutine because they received the same value x at least $n - 4f$ times. However, there is also a constant probability that the result of the shared coin toss is x !
- Of course, all nodes must take part in the execution of the subroutine
- This randomized algorithm terminates in a constant number of rounds (in expectation)!

Randomized Algorithm: Summary

The randomized algorithm has several advantages:

+ It only takes a constant number of rounds in expectation
+ It can handle crash failures even if communication is asynchronous

However, it also has some disadvantages:

– It works only if there are $f < n/9$ crash failures. It doesn't work if there are Byzantine nodes
– It only works for binary input

There are similar algorithms for the shared memory model

Can it be improved?

+ There is a constant expected time algorithm that tolerates $f < n/2$ crash failures
• What about Byzantine failures?

Byzantine **and** Asynchronous?

- Are there algorithms that can solve Byzantine agreement in an asynchronous environment (faster than simple individual coin flips)?

\+ Yes, there are.

– However, the solution comes with a cost. The fraction of nodes that can be Byzantine decreases to $1/35715$. Also, the expected message complexity of the state of the art algorithm is $O(n^3)$.

\+ Message sizes remain polynomial.

\+ The algorithm does not use a global coin, but tries to detect (biased) local coinflips from Byzantine processes.

Message Complexity

- In all of the previously discussed message passing algorithms, each (good) node broadcasts. Message complexity is $\Omega(n^2)$.

- It was believed for long that the communication overhead of Byzantine Agreement is inherently large.

- Can the message complexity be improved?

+ Yes!

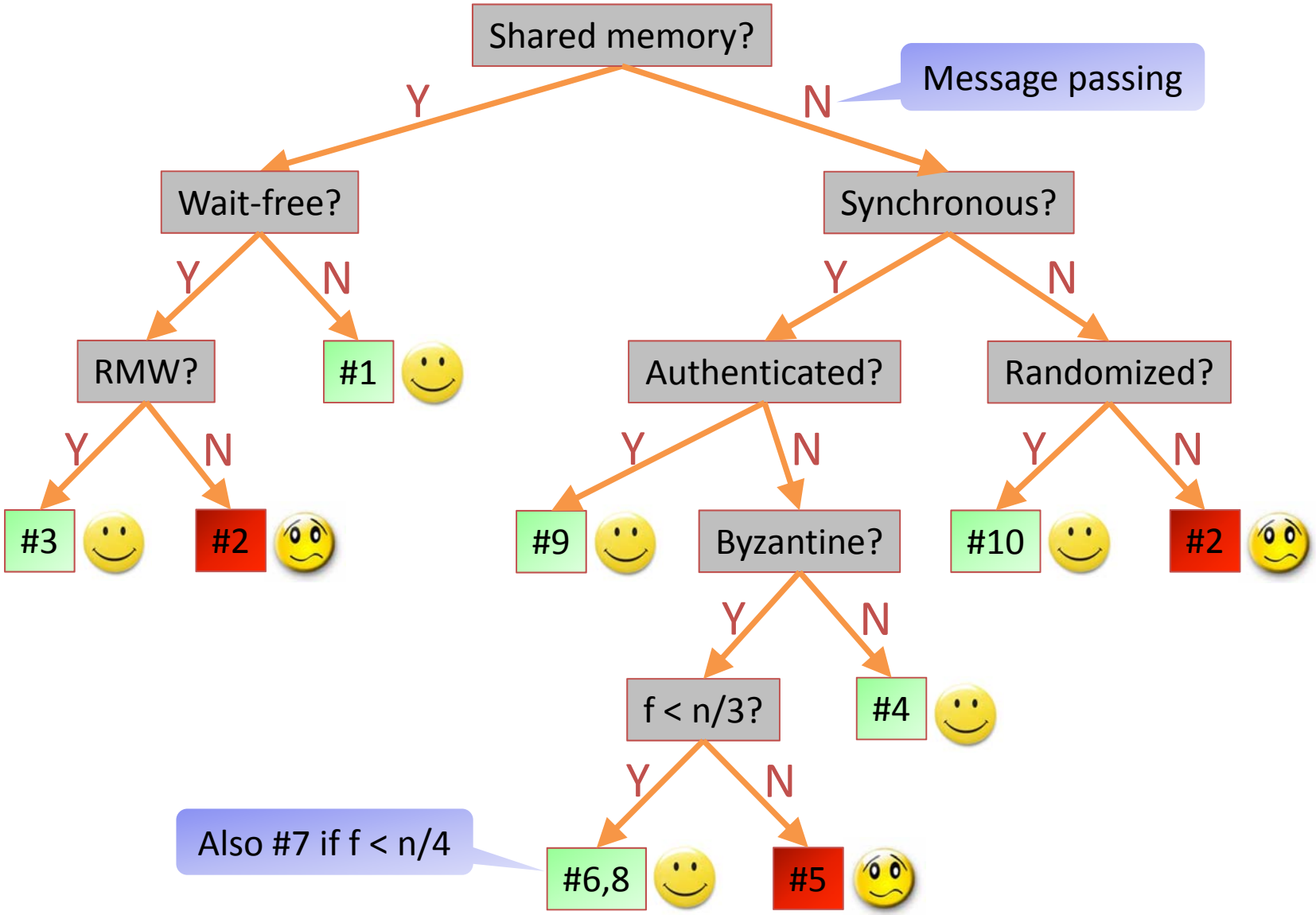
+ Byzantine Agreement can be solved in a synchronous environment with $\tilde{O}(n\sqrt{n})$ messages.

– Requires polylogarithmic time.

Summary

- We have solved consensus in a variety of models
- In particular we have seen
  - algorithms
  - wrong algorithms
  - lower bounds
  - impossibility results
  - reductions
  - etc.

Consensus: Decision Tree



Credits

- The impossibility result (#2) is from Fischer, Lynch, Patterson, 1985
- The hierarchy (#3) is from Herlihy, 1991.
- The synchronous studies (#4) are from Dolev and Strong, 1983, and others.
- The Byzantine agreement problem (#5) and the simple algorithm (#6) are from Lamport, Shostak, Pease, 1980ff., and others
- The Queen algorithm (#7) and the King algorithm (#8) are from Berman, Garay, and Perry, 1989.
- The algorithm using authentication (#9) is due to Dolev and Strong, 1982.
- The first randomized algorithm (#10) is from Ben-Or, 1983.
- The concept of a shared coin was introduced by Bracha, 1984.
- Byzantine Agreement with fewer than n^2 messages is from King and Saia 2011.
- Byzantine Agreement in an asynchronous setting is from King and Saia 2013.