Albert-Ludwigs-Universität
Institut für Informatik
Prof. Dr. F. Kuhn

July 07, 2015

Network Algorithms, Summer Term 2015
Problem Set 10 – Sample Solution

Exercise 1: Determining the Median

As stated in the hint, we start with initializing the nodes to give them IDs from $1, \dots, n$. Now the node with ID i transmits its token in time slot i. Each node uses two variables to count the number of tokens which were transmitted with a higher or lower number. After the execution each node knows how many tokens were larger/smaller than its own. Thus, the node whose token is the median can simply transmit it afterwards.
Let us briefly analyze the time and space used by this. We know from the lecture that initialization takes $\mathcal{O}(n)$ rounds. The sending of each token afterwards takes exactly n rounds, i.e., does not increase the asymptotic runtime.
Each node needs to store a new ID of up to n, i.e., needs $\mathcal{O}(\log n)$ space. The variables from the algorithm can also be stored by using $\mathcal{O}(\log n)$. After the initialization, we require that each node keeps two counters to count how many numbers were larger/smaller than our own. But since we do not need to store the values, this requires only $\log n$ space.
The correctness follows by the construction of the algorithm. The unique node whose upper and lower counter has the same value, can broadcast it and thus all nodes are aware of it.

Exercise 2: Finding Maximum

```
while TRUE do
  elect leader
  leader broadcasts value
  if If own value is bigger then
    broadcast own value
  end if
  if no transmitter then
    leader has max
  else if single transmitter then
    transmitter has max
  else if own value ≤ leader then
    exit
  end if
end while
```

A round is good if at least half the vertices exit in that round. At most $\log(n)$ good rounds are needed to find the maximum. Since the leaders are chosen randomly and independently in each round, a round is good with probability at least $\frac{1}{2}$. Let the random variable G be the number of good rounds after $4c \cdot \log(n)$ rounds. The expectation of G is bound by

$$E[G] \geq \frac{1}{2} 4c \cdot \log(n) = 2c \cdot \log(n) \geq 2 \cdot \log(n)$$

Using the Chernoff bound and setting $\delta = \frac{1}{2}$ we get the following as probability of failure

$$\begin{aligned} Pr[G < \log(n)] &= Pr[G < (1-\delta)2 \cdot \log(n)] \\ &\leq Pr[G < (1-\delta)E[G]] \\ &\leq e^{\frac{\delta^2}{2}E[G]} \\ &= e^{-\frac{1}{4}c\log(n)} \\ &= n^{-\frac{c}{4}} \end{aligned}$$

Additionally we have a probability of failure of $n^{-c'}$ in every leader election. Setting $c' = \frac{c}{4}$ and using the union bound we get

$$\begin{aligned} & Pr[\text{fail}] = P\left[\{G < \log(n)\} \cup \bigcup_i \{\text{leader election in round } i \text{ failed}\}\right] \\ & \leq (1 + c \cdot \log(n))n^{-c'} \\ & \leq 2c \cdot \log(n)n^{-c'} && \text{ignore } n = 1 \text{ because } Pr \leq 1^{-c} \\ & < 2^{\log(2c)\cdot\log(n)}n^{-c'} \\ & = n^{-c'+\log(2c)} \\ & = n^{-\alpha} \end{aligned}$$

where α is independent of n and can be set arbitrarily high by choosing the constant c'.