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Network Algorithms, Summer Term 2015
Problem Set 7 – Sample Solution

Exercise 1: Coloring Rings

1. Let $n \geq 4$ be even, and $r = n/2 - 2$. Consider the r-neighborhood graph $\mathcal{N}_r(R_n)$ of the ring R_n with n nodes. Note that for $r = n/2 - 2$ the r-neighborhood of a node contains all but three identifiers, ordered according to their occurrence.

   Then it follows from Lemma 7.5 that the ring can be colored legally with two colors in r rounds if and only if $\mathcal{N}_r(R_n)$ is bipartite, i.e., the r-neighborhood contains no odd cycle. However, there is one of length $n - 1$:
   $(1, \ldots, n-3), (2, \ldots, n-2), (3, \ldots, n-1), (4, \ldots, n), (5, \ldots, n, 1), \ldots,$
   $(n, 1, 2 \ldots, n-4), (1, \ldots, n-3)$.
   Thus no coloring of the ring with 2 colors is possible in less than $n/2 - 1$ rounds.

2. Each node informs its two neighbors whether it is in the MIS or not and additionally sends its identifier. If node v is in the MIS, it sets its color to 1. If v is not in the MIS but both of its neighbors are, then v sets its color to 2. If v has a neighbor w not in the MIS, v chooses color 2 if its identifier is larger than w's identifier, otherwise v chooses the color 3.

   The algorithm only needs one communication round. Correctness follows from the fact that either a node v is in the MIS or at least one of its neighbors is. Thus, a MIS can at best be computed one round faster than a 3-coloring, which implies that computing a MIS costs at least $(\log^* n)/2 - 2$ rounds (since coloring a directed ring with 3 or less colors needs at least $(\log^* n)/2 - 1$ rounds. See Theorem 7.11).