Albert-Ludwigs-Universität, Inst. für Informatik
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Theoretical Computer Science - Bridging Course
Winter Term 2016
Exercise Sheet 11

Hand in (electronically or hard copy) before your weekly meeting but not later than 23:59, Wednesday, January 25, 2016

Exercise 1: Inference Rules and Calculi (3+3 points)

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* **C** is described by a *set* of inference rules.
Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus **C** to 'generate' new formulae until ψ is obtained.

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C_1}: \quad \frac{\varphi \to \psi, \psi \to \chi}{\varphi \to \chi}, \frac{\neg\varphi \to \neg\psi}{\psi \to \varphi}, \frac{\varphi \to \psi, \psi \to \varphi}{\varphi \leftrightarrow \psi}$$

$$\mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi}, \frac{\varphi \wedge \psi}{\varphi, \psi}, \frac{\varphi, \psi}{\varphi \wedge \psi}, \frac{\neg(\varphi \vee \psi)}{\neg\varphi \wedge \neg\psi}, \frac{\neg\neg\varphi}{\varphi}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \to r, \neg p \to \neg q, \neg q \to \neg r\} \vdash_{\mathbf{C_1}} p \leftrightarrow q$

(b) $\{\neg(\neg p \vee q), \neg q \to (r \vee s), (r \vee s) \to t\} \vdash_{\mathbf{C_2}} t \wedge p$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.

Exercise 2: Completeness and Correctness of Calculi (3+2+2 points)

A calculus **C** is called *correct* if for every knowledge base KB and formula φ the following holds

$$KB \vdash_{\mathbf{C}} \varphi \quad \Longrightarrow \quad KB \models \varphi.$$

Calculus **C** is called *complete* if

$$KB \models \varphi \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \varphi.$$

Remark: For the definition of '$\models$' consult Exercise Sheet 10 or the lecture.

In addition to the calculi from Exercise 1 consider the calculus

$$\mathbf{C_3}: \quad \frac{\varphi, \psi \to \varphi}{\psi}$$

(a) Show that the first rule of both $\mathbf{C_1}$ and $\mathbf{C_2}$ are correct. *Hint: Use truth tables.* Give a short explanation why $\mathbf{C_1}, \mathbf{C_2}$ are correct.

(b) Show that $\mathbf{C_3}$ is not correct. *Hint: Use a truth table*

(c) Show that $\mathbf{C_1}, \mathbf{C_2}, \mathbf{C_3}$ are not complete by giving a knowledge base KB and a formula φ such that $KB \models \varphi$ but not $KB \vdash_{\mathbf{C_i}} \varphi$.

Exercise 3: Resolution (2+2+3+0 points)

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base KB and formula φ it holds

$$KB \models \varphi \quad \Longleftrightarrow \quad KB \cup \{\neg\varphi\} \models \bot.$$

Remark: $\bot$ is a formula that is unsatisfiable.

Thus, in order to show that KB entails φ, we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus $\mathbf{C}$ is called *refutation-complete* if for every knowledge base KB

$$KB \models \bot \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \bot.$$

Therefore, if we have a refutation-complete calculus $\mathbf{C}$, it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \bot$ in order to prove $KB \models \varphi$.

The *Resolution Calculus*[1] $\mathbf{R}$ is refutation-complete for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base KB is in CNF if it is of the form $KB = \{C_1, \dots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \dots, L_{i,m_i}\}$ each consist of m_i literals $L_{i,j}$
Remark: KB represents the formula $C_1 \wedge \dots \wedge C_n$ with $C_i = L_{i,1} \vee \dots \vee L_{i,m_i}$.

The Resolution Calculus has only one inference rule, the *resolution rule*:

$$\mathbf{R}: \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

Remark: L is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in KB (C_1, C_2 may be empty). To show $KB \vdash_{\mathbf{R}} \bot$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses L and $\neg L$. These entail the empty clause (defined as $\Box$), i.e. a contradiction ($\{L, \neg L\} \vdash_{\mathbf{R}} \bot$).

(a) We want to show $\{p \vee q, q \rightarrow (r \wedge s), (p \vee r) \rightarrow u\} \models u$. First convert this problem instance into a form that can be solved via resolution as described above. Document your steps.

(b) Now, use resolution to show $\{p \vee q, q \rightarrow (r \wedge s), (p \vee r) \rightarrow u\} \models u$.

(c) Using resolution, show that $(p \wedge q) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg r) \vee (q \wedge r) \vee (p \wedge \neg q)$ is a tautology.
Hint: φ tautology $\Leftrightarrow \top \models \varphi \Leftrightarrow \neg\varphi \models \bot \Leftrightarrow \neg\varphi$ unsatisfiable .

(d) Assuming $\mathcal{P} \neq \mathcal{NP}$, argue why proving logical entailment via resolution as described above, can not be done in polynomial time. *(voluntary)*

[1]Complete calculi are unpractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.