Albert-Ludwigs-Universität, Inst. für Informatik
Prof. Dr. Fabian Kuhn
M. Ahmadi, P. Bamberger, H. Ghodselahi, Y. Maus, P. Schneider

# Theoretical Computer Science - Bridging Course
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# Exercise Sheet 10 - Sample Solution

## Exercise 1: Completeness and Correctness of Calculi (2+1+1 points)

A calculus $\mathbf{C}$ is called *correct* if for every knowledge base $KB$ and formula $\varphi$ the following holds

$$KB \vdash_{\mathbf{C}} \varphi \quad \Longrightarrow \quad KB \models \varphi.$$

Calculus $\mathbf{C}$ is called *complete* if

$$KB \models \varphi \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \varphi.$$

*Remark: For the definition of '$\models$' consult Exercise Sheet* 9 *or the lecture.*

Consider the following calculuses

$$\mathbf{C_1}: \quad \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi} \qquad \mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi} \qquad \mathbf{C_3}: \quad \frac{\varphi, \psi \to \varphi}{\psi}$$

(a) Show that the $\mathbf{C_1}$ and $\mathbf{C_2}$ are both correct. *Hint: Use truth tables.* Give a short explanation why $\mathbf{C_1}, \mathbf{C_2}$ are correct.

(b) Show that $\mathbf{C_3}$ is not correct. *Hint: Use a truth table*

(c) Show that $\mathbf{C_1}, \mathbf{C_2}, \mathbf{C_3}$ are not complete by giving a knowledge base $KB$ and a formula $\varphi$ such that $KB \models \varphi$ but not $KB \vdash_{\mathbf{C_i}} \varphi$.

## Sample Solution

(a) The inference rules we used are known logical entailments from the lecture which is why they are correct. We prove this for $\mathbf{C_1}, \mathbf{C_2}$ with a truth table by checking whether every model of the formulae in the premise is also a model of the inferred formulae.

| model of $KB$ | $\varphi$ | $\psi$ | $\varphi \leftrightarrow \psi$ | $\varphi \to \chi$ | $\psi \to \chi$ |
|---|---|---|---|---|---|
| ✓ | 0 | 0 | **1** | **1** | **1** |
| | 0 | 1 | 0 | 1 | 0 |
| | 1 | 0 | 0 | 0 | 1 |
| ✓ | 1 | 1 | **1** | **1** | **1** |

| model of $KB$ | $\varphi$ | $\psi$ | $\varphi \to \psi$ |
|---|---|---|---|
| | 0 | 0 | 1 |
| | 0 | 1 | 1 |
| | 1 | 0 | 0 |
| ✓ | **1** | **1** | **1** |

(b) Consider $\varphi, \psi$ and an interpretation $I$, such that $I(\varphi) = T$ and $I(\psi) = F$. Then $I$ is a model of the premise since $I(\varphi) = T, I(\psi \to \varphi) = T$ but not of the inferred formula since $I(\psi) = F$.

(c) Inference rules of a calculus can only accomplish syntactical replacement much like a rules of a grammar and carry no semantics. This means e.g., that if a symbol does occur neither in the knowledge base nor in the inference rules, we can never derive such a symbol.

Thus we have $p \rightarrow q \models \neg p \vee q$ but $p \rightarrow q \nvdash_{\mathbf{C_1}} \neg p \vee q$.

Another way to argue, is that from an unsatisfiable formula $\bot$ we can derive everything by definition: $\bot \models \varphi$ since all models of $\bot$ (i.e., none) are models of $\varphi$. However we have no such inference rule in $\mathbf{C_2}$ and $\mathbf{C_3}$. Thus for an arbitrary $\varphi$ we have $\bot \models \varphi$ but $\bot \nvdash_{\mathbf{C_2}} \varphi$, $\bot \nvdash_{\mathbf{C_3}} \varphi$.

## Exercise 2: Resolution (1+2+3 points)

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base $KB$ and formula $\varphi$ it holds

$$KB \models \varphi \iff KB \cup \{\neg\varphi\} \models \bot.$$

*Remark: $\bot$ is a formula that is unsatisfiable.*

Thus, in order to show that $KB$ entails $\varphi$, we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus **C** is called *refutation-complete* if for every knowledge base $KB$

$$KB \models \bot \implies KB \vdash_{\mathbf{C}} \bot.$$

Therefore, if we have a refutation-complete calculus **C**, it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \bot$ in order to prove $KB \models \varphi$.

The *Resolution Calculus*[1] **R** is correct and refutation-complete for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base $KB$ is in CNF if it is of the form $KB = \{C_1, \ldots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \ldots, L_{i,m_i}\}$ each consist of $m_i$ literals $L_{i,j}$
*Remark: $KB$ represents the formula $C_1 \wedge \ldots \wedge C_n$ with $C_i = L_{i,1} \vee \ldots \vee L_{i,m_i}$.*

The Resolution Calculus has only one inference rule, the *resolution rule*:

$$\mathbf{R}: \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

*Remark: $L$ is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in $KB$ ($C_1, C_2$ may be empty). To show $KB \vdash_{\mathbf{R}} \bot$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses $L$ and $\neg L$. These entail the empty clause (defined as $\Box$), i.e. a contradiction ( $\{L, \neg L\} \vdash_{\mathbf{R}} \bot$ ).*

(a) We want to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$. First convert this problem instance into a form that can be solved via resolution as described above. Document your steps.

(b) Now, use resolution to show $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$.

(c) Consider the sentence "Heads, I win". "Tails, you lose". Design a propositional $KB$ that represents these sentences (create the propositions and rules required). Then use propositional resolution to prove that **I always win**.

## Sample Solution

(a) We transform $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u\} \models u$ into the form $KB \models \bot$ where $KB$ is in CNF. The given entailment is equivalent to $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u, \neg u\} \models \bot$ using the Contraposition Theorem, which we described above. Now we transform the knowledge base into CNF using DeMorgan's rule and distribution among others.

$$\begin{aligned} & \{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow u, \neg u\} \\ \equiv\ & \{p, q, \neg p \vee r, \neg(q \wedge r) \vee u, \neg u\} \\ \equiv\ & \{p, q, \neg p \vee r, \neg q \vee \neg r \vee u, \neg u\} \\ \equiv\ & \{\{p\}, \{q\}, \{\neg p, r\}, \{\neg q, \neg r, u\}, \{\neg u\}\} \end{aligned}$$

[1] Complete calculi are unpractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.

(b) Now we can use the Resolution calculus $\mathbf{R}$ to derive a contradiction (the empty clause $\Box$).

$$\begin{aligned} \{\neg p, r\}, \{p\} &\quad \vdash_{\mathbf{R}} \quad \{r\} \\ \{\neg q, \neg r, u\}, \{r\} &\quad \vdash_{\mathbf{R}} \quad \{\neg q, u\} \\ \{\neg q, u\}, \{u\} &\quad \vdash_{\mathbf{R}} \quad \{\neg q\} \\ \{\neg q\}, \{q\} &\quad \vdash_{\mathbf{R}} \quad \Box \end{aligned}$$

Since $\mathbf{R}$ is correct, this proves the original entailment.

(c) 1) Make our objects: $H$ : heads $\quad T$ : tails $\quad I$ : I win $\quad Y$ : You win.

2) State your rules: $H \to I$ and $T \to \neg Y$.

3) We now must specify implicit rules. The system does not know that heads and tails are mutually exclusive: $H \otimes T$ and $I \otimes Y$.

4) Convert to CNF: $\neg H \vee I \quad \neg T \vee \neg Y \quad (H \vee T) \wedge (\neg H \vee \neg T) \quad (I \vee Y) \wedge (\neg I \vee \neg Y)$.

5) We want to prove $I$, hence we insert the literal $\neg I$ for the proof by contradiction. Now we start resolving clauses:

- $\neg T \vee \neg Y$ and $H \vee T$ resolves to $H \vee \neg Y$.
- $\neg H \vee I$ and $H \vee \neg Y$ resolves to $I \vee \neg Y$.
- $\neg I$ and $I \vee \neg Y$ resolves to $\neg Y$.
- $I \vee Y$ and $\neg Y$ resolves to $I$.
- $I$ and $\neg I$ resolves to $\Box$.

Consequently, we have a *contradiction*. Thus, $I$ is true.

## Exercise 3: Predicate Logic: Construct Formulae (1+1+1+1 points)

Let $\mathcal{S} = \langle \{x, y, z\}, \emptyset, \emptyset, \{R\} \rangle$ be a signature. Translate the following sentences of first order formula over $\mathcal{S}$ into idiomatic English. Use $R(x, y)$ as statement "$x$ is a part of $y$".

(a) $\exists x \forall y R(x, y)$.

(b) $\exists y \forall x R(x, y)$.

(c) $\forall x \forall y \exists z (R(x, z) \land R(y, z))$

(d) $\forall x \exists y (R(y, x) \land \neg \exists z (R(z, y) \land \neg R(y, z)))$

## Sample Solution

(a) Something is a part of everything.

(b) Something has everything as a part.

(c) For any two things, there is something of which they are both a part.

(d) Everything has a part which has no part of which it is not also a part.

## Exercise 4: Predicate Logic: Entailment (2+2+2 points)

Let $\varphi, \psi$ be first order formulae over signature $\mathcal{S}$. Similar to propositional logic, in predicate logic we write $\varphi \models \psi$ if every model of $\varphi$ is also a model for $\psi$. We write $\varphi \equiv \psi$ if both $\varphi \models \psi$ and $\psi \models \varphi$. A *knowledge base* $KB$ is a set of formulae. A model of $KB$ is model for all formulae in $KB$. We write $KB \models \varphi$ if all models of $KB$ are models of $\varphi$. Show or disprove the following entailments.

(a) $(\exists x\, R(x)) \wedge (\exists x\, P(x)) \wedge (\exists x\, T(x)) \models \exists x\, (R(x) \wedge P(x) \wedge T(x))$.

(b) $(\forall x \forall y\, f(x,y) \doteq f(y,x)) \wedge (\forall x\, f(x, \mathbf{c}) \doteq x) \models \forall x\, f(\mathbf{c}, x) \doteq x$.

(c) $(\forall x\, R(x,x)) \wedge (\forall x \forall y\, R(x,y) \wedge R(y,x) \rightarrow x \doteq y) \wedge (\forall x \forall y \forall z\, R(x,y) \wedge R(y,z) \rightarrow R(x,z))$
$\models \forall x \forall y\, R(x,y) \vee R(y,x)$.

*Hint: Consider order relations. E.g., $a \leq b$ (a less-equal b) and $a|b$ (a divides b).*

## Sample Solution

(a) The stated entailment is false (it holds in the other direction though). In order to disprove it, we give a model for the left side which is not a model for the right side.

Let $I = \langle \{a,b\}, \cdot^I \rangle$ with $R^I = \{a\}$, $P^I = \{b\}$, and $T^I = \{a\}$. This makes the left side true since there exists an element $x = a$ that makes $R(x)$ and $T(x)$ true and an element $x = b$ that makes $P(x)$ true (Note the brackets around the three $\exists$ quantifiers which mean that the three elements need not necessarily be the same).

However $R(a) \wedge P(a) \wedge T(a) = T \wedge F \wedge T = F$ and $R(b) \wedge P(b) \wedge T(b) = F \wedge T \wedge F = F$ thus the right side is false (there exists no element which makes the three relations' symbols $R, P, T$ true, since we tested all that are in the domain).

(b) The stated entailment holds. We prove this by picking an arbitrary *model* (!) $I = \langle \mathcal{D}, \cdot^I \rangle$ of the left-hand formula. We show that $I$ is a model for the right-hand formula, too. For that purpose let $x$ be an arbitrary element from $\mathcal{D}$.

Since $I$ is a model for the left side we already know $f(x, \mathbf{c}^I) \doteq x$. The first condition in the left formula encodes the commutative property. Since $\mathbf{c}^I$ is also an element from the domain $\mathcal{D}$ we know $f(x, \mathbf{c}) = f(\mathbf{c}^I, x)$ and thus $f(\mathbf{c}^I, x) \doteq x$. Since $x$ was arbitrary we have $\forall x\, f(\mathbf{c}^I, x) \doteq x$.

(c) The formula on the left side encodes the properties of an *order relation.* The formula on the right side encodes the property of *totality* of an order, which means that every element is related to (read: can be compared with) every other element. However, in general an order relation does not need to be total (which is called a *partial order*).

The hint proposes two order relations, one of which is total over the domain of integers $\mathbb{Z}$ (either '$x \leq y$' or '$x \leq y$' or both) whereas the other is not (it may happen that neither $x|y$ nor $x|y$). Thus the logical entailment is false since with $\mathbb{Z}$ and the 'divides'-relation we have a model of the left-hand formula which is no model of the right-hand one (it is not total).

We formalize this as follows. Let $I = \langle \mathbb{Z}, \cdot^I \rangle$ with $R^I := \{(x,y) \in \mathbb{Z} \mid x \text{ divides } y\}$. Obviously we have the reflexive property since $x \in \mathbb{Z}$ divides itself. If $x \in \mathbb{Z}$ divides $y \in \mathbb{Z}$ and $y \in \mathbb{Z}$ divides $z \in \mathbb{Z}$ then $x$ also divides $z$ which gives us transitivity. Finally, if $x$ divides $y$ and vice versa then $y$ is multiple of $x$ and vice versa which means that the multiplicand must in both cases be 1 thus both $x$ and $y$ are equal which gives us the antisymmetry property.

This means that $I$ is a model of the left-hand formula. Now consider the two primes $x = 2$ and $y = 3$. By definition of prime numbers neither of the two can divide the other. Thus $\forall x \forall y\, R(x,y) \vee R(y,x)$ is false. Therefore $I$ can be no model of the right-hand formula.