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Advanced Algorithms
Sample Solution Problem Set 7

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Exercise 1: Maximum Flow as Zero Sum Game

Let $G = (V, E)$ be a graph with edge capacities $c : E \to \{1\}$ and let $s, t \in V$ be the source and sink respectively. We can formulate the maximum flow problem as zero sum game as follows. Let P be the (rather large) set of s-t paths. We define a path player $\mathcal{P}$ which picks a path from P and an edge player $\mathcal{E}$ which picks an edge from E. Let $e \in E$ and $p \in P$ be the choices of $\mathcal{E}$ and $\mathcal{P}$. Then the payoff for $\mathcal{E}$ is 1 if $e \in p$, else 0. Show that the value of this game is $\frac{1}{\gamma}$ where γ is the value of the maximum s-t flow.

Sample Solution

Our goal is to show: (i) player $\mathcal{P}$ has a strategy that guarantees it a loss of at most $1/\gamma$ and (ii) player $\mathcal{E}$ has a strategy that guarantees it a win of at least $1/\gamma$. Given that (i),(ii) are true, the value of the game must obviously be $1/\gamma$.
Let f be a maximum flow on G. We use f to show (i). Let $f(p)$ be the flow routed over some path $p \in P$. We define the probability that player $\mathcal{P}$ picks path $p \in P$ as $q_p := \frac{f(p)}{\gamma}$. Now assume player $\mathcal{E}$ picks some edge $e \in E$. Then the probability that e is on a path that player $\mathcal{P}$ picks is

$$\sum_{p \ni e} q_p = \sum_{p \ni e} \frac{f(p)}{\gamma} \overset{c(e)=1}{\leq} \frac{1}{\gamma}.$$

To show (ii) we will formulate the max flow problem as a primal-dual pair of linear programs, with the goal of using strong duality to get a good strategy for player $\mathcal{E}$. For each $p \in P$ we define a variable x_p. Then max flow corresponds to the solution of the (large) LP:

$$\begin{aligned} &\max \sum_{p \in P} x_p \\ \text{s.t.} \quad &\forall e \in E : \sum_{p \ni e} x_p \leq 1 \\ &\forall p \in P : \; x_p \geq 0 \end{aligned}$$

An optimal solution x_p^*, $p \in P$ corresponds to a max flow, hence $\sum_{p \in P} x_p^* = \gamma$. For the dual we define a variable x_e for each $e \in E$. We have

$$\begin{aligned} &\min \sum_{e \in E} x_e \\ \text{s.t.} \quad &\forall p \in P : \sum_{e \in p} x_e \geq 1 \\ &\forall e \in E : \; x_e \geq 0 \end{aligned}$$

Since the primal is feasible and bounded (by γ), strong duality tells us that the optimal solution x_e^*, $e \in E$ of the dual also has optimal value $\sum_{e\in E} x_e^* = \gamma$. Now we show that $\mathcal{E}$ can get an optimal strategy out of the optimal solution to the above dual LP. Let $\mathcal{E}$ pick an edge with probability $q_e := \frac{x_e^*}{\sum_{e\in E} x_e^*}$ Then for any path p player $\mathcal{P}$ might choose we have

$$\sum_{e\in p} q_e \geq \frac{\sum_{e\in p} x_e^*}{\sum_{e\in E} x_e^*} \geq \frac{1}{\sum_{e\in E} x_e^*} = \frac{1}{\gamma}.$$

Exercise 2: Maximum Flow with Multiplicative Weight Updates

Let $G = (V, E)$ be a graph with edge capacities $c : E \to \{1\}$ and let $s, t \in V$ be the source and sink respectively. Assume the value γ of the maximum s-t flow is given. Use MWU to efficiently compute a s-t flow f of total size γ, that has at most $(1+\varepsilon)$ flow per edge.

Sample Solution

Roughly speaking, our strategy is to let players $\mathcal{P}$ and $\mathcal{E}$ play MWU against each other with respect to the zero sum game defined above (c.f. Exercise 1). More precisely, as $\mathcal{P}$ has a huge set of options to choose from (number of s-t-paths is potentially huge), we let only $\mathcal{E}$ play MWU and let player $\mathcal{P}$ respond each round in some more efficient manner.

We do this for a total of $T := \frac{4\gamma^2 \ln n}{\varepsilon^2}$ rounds where player $\mathcal{E}$ will compute distributions $\boldsymbol{q}^1, \dots, \boldsymbol{q}^T$ with respect to some – yet to be defined – gain-function g_e^t. In step t we let player $\mathcal{P}$ compute a good path $p^t \in P$ in response to the distribution $\boldsymbol{q}^t$ (one that minimizes $\mathcal{P}$'s losses), from which we can define g_e^{t+1} for the next round. Our hope is that the paths $p_1, \dots, p_T$ that player $\mathcal{P}$ computes can be used to construct a good s-t-flow.

Initially, player $\mathcal{E}$ starts with the $\boldsymbol{w}^0 = (1, \dots, 1)$ vector (of dimension $|E|$) representing the uniform distribution $\boldsymbol{q}^0$ on edges (obtained by normalizing $\boldsymbol{w}^0$). In step $t \geq 1$ let $\boldsymbol{q}^{t-1}$ be the distribution from the previous step. Then the optimal response of player $\mathcal{P}$ is to choose the path p_t that minimizes $\sum_{e\in p} q_e^{t-1}$, i.e. the chance that player $\mathcal{E}$ chooses an edge on p_t. This can be done by computing a *shortest path* in G with edge weights q_e^{t-1}. Subsequently, for each $e \in E$, we define the gain

$$g_e^t := \begin{cases} 1, & e \in p_t \\ 0, & \text{else} \end{cases}$$

and update our weights $\boldsymbol{w}^t$ accordingly to obtain the next distribution $\boldsymbol{q}^t$ on E. We obtain a result by taking the flow f that routes γ/T units of flow over each path p_t. As pseudocode:

Algorithm 1 ApproximateFlow($G = (V, E), \gamma, \varepsilon$)

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q^0 ← (1/|E|, ..., 1/|E|)                    ▷ vector that represents uniform distribution on E
for t = 1 to T ← 4γ² ln n / ε² do
    p_t ← arg min_{p∈P} Σ_{e∈p} q_e^{t-1}     ▷ compute shortest path on G with edge weights q_e^{t-1}
    for each e ∈ E do                          ▷ compute new gains g_e^t
        if e ∈ p_t then g_e^t ← 1
        else g_e^t ← 0
    q^t ← new distribution based on g_e^t      ▷ determined according to the MWU scheme
return flow f that routes γ/T units of flow over each path p_1, ..., p_T
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Obviously f has value γ. It remains to prove that f routes at most $1+\varepsilon$ units of flow over each edge. For a contradiction assume there is some edge $e \in E$ with more than $1+\varepsilon$ flow. Then more than $(1+\varepsilon)T/\gamma$ paths of the selected ones $p_1, \dots, p_T$ use e. Then, in hindsight, if player $\mathcal{E}$ would have played

e all the time, he would have gained strictly more than $(1+\varepsilon)T/\gamma$ (thus the best edge $e^* \in E$ that player $\mathcal{E}$ can choose is at least as good).
Now consider a maximum flow f^* of G. We can assume that the flow over each path that f^* induces is integral since all edge weights are. Since f^* has value γ and since the edge capacities are 1, there are γ paths that use their edges exclusively. Therefore player $\mathcal{E}$ can assign at most $1/\gamma$ total probability to each of the paths *on average*. Since player $\mathcal{P}$ will pick the shortest path in terms of total probability it will pick one that has at most $1/\gamma$ (no matter how the edge player assigns probabilities $\boldsymbol{q}^t$). Therefore player $\mathcal{E}$ can win at most $1/\gamma$ each round in expectation, which is at most T/γ in total. Hence, due our choice of $T = \frac{4\gamma^2 \ln n}{\varepsilon^2}$, we obtain

$$\text{regret} > \frac{(1+\varepsilon)T}{\gamma} - \frac{T}{\gamma} = \frac{\varepsilon T}{\gamma} = \frac{4\gamma \ln n}{\varepsilon}.$$

However, from the guarantees of MWU we know that the procedure has a regret of at most

$$\text{regret} \leq 2\sqrt{T \ln n} = 2\sqrt{\frac{4\gamma^2 \ln^2 n}{\varepsilon^2}} = \frac{4\gamma \ln n}{\varepsilon},$$

a contradiction.