



Algorithms and Datastructures Summer Term 2020 Sample Solution Exercise Sheet 2

Due: Wednesday, 27th of May, 4 pm.

Exercise 1: \mathcal{O} -notation

(7 Points)

Prove or disprove the following statements. Use the *set definition* of the \mathcal{O} -notation (lecture slides week 2, slides 11 and 12).

- (a) $3n^3 + 8n^2 + n \in \mathcal{O}(n^3)$ (1 Point)
- (b) $2^n \in o(n!)$ (2 Points)
- (c) $2 \log n \in \Omega((\log n)^2)$ (2 Points)
- (d) $\max\{f(n), g(n)\} \in \Theta(f(n) + g(n))$ for non-negative functions f and g . (2 Points)

Sample Solution

- (a) True. Choose $n_0 = 1$ and $c = 12$. For $n \geq n_0$ we have $n^3 \geq n^2 \geq n$ and hence $3n^3 + 8n^2 + n \leq 12n^3 = cn^3$.
- (b) True. Let $c > 0$. Choose $n_0 = \max\{\frac{1}{c}, 8\}$. For $n \geq n_0$ we have

$$c \cdot n! \stackrel{n \geq 1/c}{\geq} \frac{1}{n} \cdot n! = (n-1)! \geq (n-1) \cdot (n-2) \cdot \dots \cdot \lfloor \frac{n}{2} \rfloor \stackrel{n \geq 8}{\geq} 4^{\frac{n}{2}} = 2^n$$

- (c) False. Let $c > 0$. We have

$$\begin{aligned} 2 \log n &\geq c(\log n)^2 \\ \Leftrightarrow \frac{2}{c} &\geq \log n \\ \Leftrightarrow \frac{2}{c^{\frac{1}{2}}} &\geq n \\ \Leftrightarrow 4^{\frac{1}{c}} &\geq n \end{aligned}$$

So for a given $n_0 \geq 1$ choose $n = \max\{n_0, 4^{\frac{1}{c}}\} + 1$. For this n we have $n > n_0$ and $2 \log n < c(\log n)^2$.

- (d) True. Choose $n_0 = 1$, $c_1 = \frac{1}{2}$ and $c_2 = 1$. For $n \geq n_0$ we have

$$c_1 \cdot (f(n) + g(n)) \leq \max\{f(n), g(n)\} \stackrel{f, g \geq 0}{\leq} c_2 \cdot (f(n) + g(n))$$

Exercise 2: Sorting by asymptotic growth

(6 Points)

Sort the following functions by their asymptotic growth. Write $g <_{\mathcal{O}} f$ if $g \in \mathcal{O}(f)$ and $f \notin \mathcal{O}(g)$. Write $g =_{\mathcal{O}} f$ if $f \in \mathcal{O}(g)$ and $g \in \mathcal{O}(f)$ (no proof needed).

\sqrt{n}	2^n	$n!$	$\log(n^3)$
3^n	n^{100}	$\log(\sqrt{n})$	$(\log n)^2$
$\log n$	$10^{100}n$	$(n+1)!$	$n \log n$
$2^{(n^2)}$	n^n	$\sqrt{\log n}$	$(2^n)^2$

Sample Solution

$$\begin{aligned} \sqrt{\log n} &<_{\mathcal{O}} \log(\sqrt{n}) =_{\mathcal{O}} \log n =_{\mathcal{O}} \log(n^3) <_{\mathcal{O}} (\log n)^2 <_{\mathcal{O}} \sqrt{n} <_{\mathcal{O}} 10^{100}n <_{\mathcal{O}} n \log n \\ &<_{\mathcal{O}} n^{100} <_{\mathcal{O}} 2^n <_{\mathcal{O}} 3^n <_{\mathcal{O}} (2^n)^2 <_{\mathcal{O}} n! <_{\mathcal{O}} (n+1)! <_{\mathcal{O}} n^n <_{\mathcal{O}} 2^{(n^2)} \end{aligned}$$

Exercise 3: Stable Sorting

(7 Points)

A sorting algorithm is called stable if elements with the same key remain in the same order. E.g., assume you want to sort the following strings where the sorting key is *the first letter by alphabetic order*:

["tuv", "adr", "bbc", "tag", "taa", "abc", "sru", "bcb"]

A *stable* sorting algorithm must generate the following output:

["adr", "abc", "bbc", "bcb", "sru", "tuv", "tag", "taa"]

A sorting algorithm is not stable (with respect to the sorting key) if it outputs, e.g., the following:

["abc", "adr", "bbc", "bcb", "sru", "taa", "tag", "tuv"]

- (a) Which sorting algorithms from the lecture (except `CountingSort`) are *not* stable? Prove your statement by giving an appropriate example. (4 Points)
- (b) Describe a method to make any sorting algorithm stable, without changing the *asymptotic* runtime. Explain. (3 Points)

Sample Solution

- (a)
- Selection Sort is not stable. Consider as input the array $[x, y, z]$ with $x.key = y.key = 1$ and $z.key = 0$. In the first step, x and z are swapped, because z has the smallest key in the array. So we get $[z, y, x]$. This array will not be altered in the second step (as $y.key = x.key$), i.e., it equals the output of Selection Sort. So x and y have been swapped.
 - Quicksort is not stable. Consider as input the array $[x, y, z, w]$ with $x.key = 1$, $y.key = z.key = 2$ and $w.key = 0$ and assume x is taken as pivot. In the first divide step, y and w are swapped (i.e., we get $[x, w, z, y]$) and the array is divided into $[x, w]$ and $[z, y]$. Recursive sorting yields $[w, x]$ and $[z, y]$ and thus $[w, x, z, y]$ will be returned. So y and z have been swapped.
 - Mergesort: If you implement Mergesort according to the pseudocode on page 26 of lecture 01, Mergesort is not stable. The reason is the condition $A[i] < A[j]$ in line 7 of the code which may cause elements with the same key to change order. In the code of `MergeSort.py` from the first exercise sheet, we used the condition $A[i] \leq A[j]$, making the algorithm stable.

- (b) Add the number i to the key of the i -th element in the array (i.e., set $A[i].key = (A[i].key, i)$). Now run the given (non-stable) sorting algorithm according to the lexicographic ordering¹ on this new set of keys. That is, we sort according to the original keys and use the index in A as tie breaker.

Changing the keys takes time $O(n)$. Additionally, each comparison between two elements is prolonged by an additional $O(1)$ steps. As any sorting algorithm takes $\Omega(n)$, the asymptotic runtime does not change.

¹Let $(A, <_A)$ and $(B, <_B)$ be ordered sets. The lexicographic ordering $<_{lex}$ on $A \times B$ is defined by $(a, b) <_{lex} (a', b') :\Leftrightarrow a <_A a' \vee (a = a' \wedge b <_B b')$