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Algorithms and Data Structures
Summer Term 2021
Sample Solution Exercise Sheet 2

Exercise 1: $\mathcal{O}$-notation

Prove or disprove the following statements. Use the *set definition* of the $\mathcal{O}$-notation (lecture slides week 2, slides 11 and 12).

(a) $4n^3 + 8n^2 + n \in \mathcal{O}(n^3)$

(b) $2^n \in o(n!)$

(c) $2\log n \in \Omega((\log n)^2)$

(d) $\max\{f(n), g(n)\} \in \Theta(f(n) + g(n))$ for non-negative functions f and g.

Sample Solution

(a) True. Choose $n_0 = 1$ and $c = 13$. For $n \geq n_0$ we have $n^3 \geq n^2 \geq n$ and hence $4n^3 + 8n^2 + n \leq 13n^3 = cn^3$.

(b) True. Let $c > 0$. Choose $n_0 = \max\{\frac{1}{c}, 8\}$. For $n \geq n_0$ we have

$$c \cdot n! \overset{n \geq 1/c}{\geq} \frac{1}{n} \cdot n! = (n-1)! \geq (n-1) \cdot (n-2) \cdot \ldots \cdot \lfloor \frac{n}{2} \rfloor \overset{n \geq 8}{\geq} 4^{\frac{n}{2}} = 2^n$$

(c) False. Let $c > 0$. We have

$$\begin{array}{rccc} & 2\log n & \geq & c(\log n)^2 \\ \Leftrightarrow & 2 & \geq & c\log n \\ \Leftrightarrow & \frac{2}{c} & \geq & \log n \\ \Leftrightarrow & 4^{\frac{1}{c}} & \geq & n \end{array}$$

So for a given $n_0 \geq 1$ choose $n = \max\{n_0, 4^{\frac{1}{c}}\} + 1$. For this n we have $n > n_0$ and $2\log n < c(\log n)^2$.

(d) True. Choose $n_0 = 1$, $c_1 = \frac{1}{2}$ and $c_2 = 1$. For $n \geq n_0$ we have

$$c_1 \cdot (f(n) + g(n)) \leq \max\{f(n), g(n)\} \overset{f,g \geq 0}{\leq} c_2(f(n) + g(n))$$

Exercise 2: Sorting by asymptotic growth

Sort the following functions by their asymptotic growth. Write $g <_{\mathcal{O}} f$ if $g \in \mathcal{O}(f)$ *and* $f \notin \mathcal{O}(g)$. Write $g =_{\mathcal{O}} f$ if $f \in \mathcal{O}(g)$ *and* $g \in \mathcal{O}(f)$ (no proof needed).

$\sqrt{n}$	2^n	$n!$	$\log(n^3)$
3^n	n^{100}	$\log(\sqrt{n})$	$(\log n)^2$
$\log n$	$10^{100}n$	$(n+1)!$	$n \log n$
$2^{(n^2)}$	n^n	$\sqrt{\log n}$	$(2^n)^2$

Sample Solution

$$\sqrt{\log n} <_{\mathcal{O}} \log(\sqrt{n}) =_{\mathcal{O}} \log n =_{\mathcal{O}} \log(n^3) <_{\mathcal{O}} (\log n)^2 <_{\mathcal{O}} \sqrt{n} <_{\mathcal{O}} 10^{100} n <_{\mathcal{O}} n \log n$$
$$<_{\mathcal{O}} n^{100} <_{\mathcal{O}} 2^n <_{\mathcal{O}} 3^n <_{\mathcal{O}} (2^n)^2 <_{\mathcal{O}} n! <_{\mathcal{O}} (n+1)! <_{\mathcal{O}} n^n <_{\mathcal{O}} 2^{(n^2)}$$

Exercise 3: Stable Sorting

A sorting algorithm is called stable if elements with the same key remain in the same order. E.g., assume you want to sort the following strings where the sorting key is *the first letter by alphabetic order*:

$$[\text{“}tuv\text{”}, \text{“}adr\text{”}, \text{“}bbc\text{”}, \text{“}tag\text{”}, \text{“}taa\text{”}, \text{“}abc\text{”}, \text{“}sru\text{”}, \text{“}bcb\text{”}]$$

A *stable* sorting algorithm must generate the following output:

$$[\text{“}adr\text{”}, \text{“}abc\text{”}, \text{“}bbc\text{”}, \text{“}bcb\text{”}, \text{“}sru\text{”}, \text{“}tuv\text{”}, \text{“}tag\text{”}, \text{“}taa\text{”}]$$

A sorting algorithm is not stable (with respect to the sorting key) if it outputs, e.g., the following:

$$[\text{“}abc\text{”}, \text{“}adr\text{”}, \text{“}bbc\text{”}, \text{“}bcb\text{”}, \text{“}sru\text{”}, \text{“}taa\text{”}, \text{“}tag\text{”}, \text{“}tuv\text{”}]$$

(a) Which sorting algorithms from the lecture (except `CountingSort`) are *not* stable? Prove your statement by giving an appropriate example.

(b) Describe a method to make any sorting algorithm stable, without changing the *asymptotic* runtime. Explain.

Sample Solution

(a)
- Selection Sort is not stable. Consider as input the array $[x, y, z]$ with $x.key = y.key = 1$ and $z.key = 0$. In the first step, x and z are swapped, because z has the smallest key in the array. So we get $[z, y, x]$. This array will not be altered in the second step (as $y.key = x.key$), i.e., it equals the output of Selection Sort. So x and y have been swapped.
- Quicksort is not stable. Consider as input the array $[x, y, z, w]$ with $x.key = 1$, $y.key = z.key = 2$ and $w.key = 0$ and assume x is taken as pivot. In the first divide step, y and w are swapped (i.e., we get $[x, w, z, y]$) and the array is divided into $[x, w]$ and $[z, y]$. Recursive sorting yields $[w, x]$ and $[z, y]$ and thus $[w, x, z, y]$ will be returned. So y and z have been swapped.
- Mergesort: If you implement Mergesort according to the pseudocode on page 26 of lecture 01, Mergesort is not stable. The reason is the condition $A[i] < A[j]$ in line 7 of the code which may cause elements with the same key to change order. If we instead use the condition $A[i] \leq A[j]$, we make the algorithm stable.

(b) Add the number i to the key of the i-th element in the array (i.e., set $A[i].key = (A[i].key, i)$). Now run the given (non-stable) sorting algorithm according to the lexocographic ordering[1] on this new set of keys. That is, we sort according to the original keys and use the index in A as tie breaker.

Changing the keys takes time $O(n)$. Additionally, each comparison between two elements is prolonged by an additional $O(1)$ steps. As any sorting algorithm takes $\Omega(n)$, the asymptotic runtime does not change.

[1] Let $(A, <_A)$ and $(B, <_B)$ be ordered sets. The lexicographic ordering $<_{lex}$ on $A \times B$ is defined by $(a, b) <_{lex} (a', b') :\Leftrightarrow a <_A a' \vee (a = a' \wedge b <_B b')$

Exercise 4: Running time

Give an asymptotically tight upper bound for the running time of the following algorithm as a function of n.

```
s ← 0
for i = 1 to n do
    j = 1
    while j < i do
        s ← s + i · j
        j ← 2 · j
```

Sample Solution

For each i, the running time of the internal while is proportional to $\log_2 i$. Hence, the total running time is proportional to $\sum_{i=1}^{n} \log_2 i$. For an upper bound, note that this sum is upper bounded by $\sum_{i=1}^{n} \log_2 n = n \log_2 n = O(n \log n)$. In order to show that it is tight, note that the sum is lower bounded by $\sum_{i=n/2}^{n} \log_2(n/2) = (n/2)\log(n/2) = \Omega(n \log n)$. Hence the running time is $\Theta(n \log n)$.