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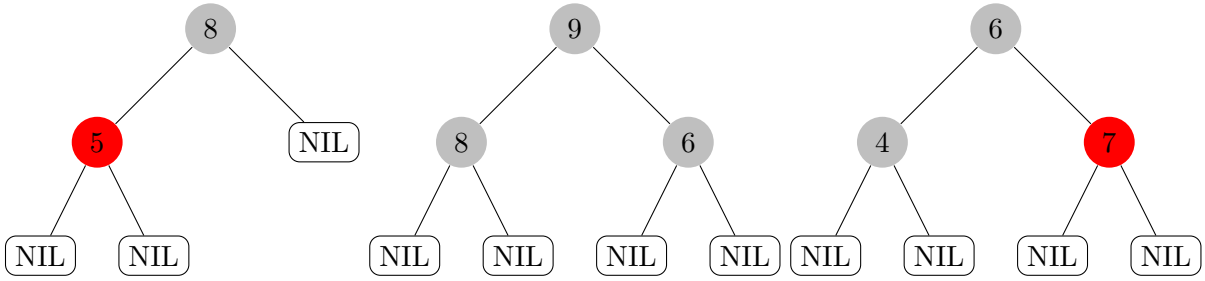
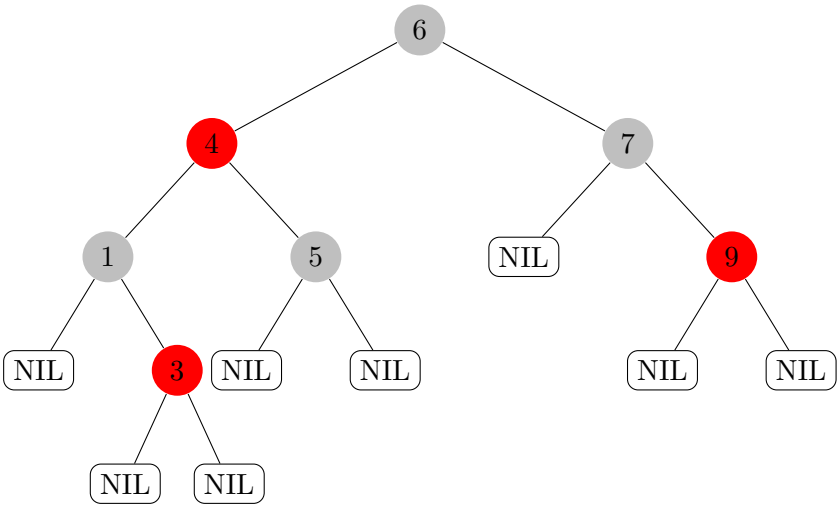
# Algorithms and Data Structures
# Summer Term 2021
# Sample Solution Exercise Sheet 7

## Exercise 1: Red-Black Trees

(a) Decide for each of the following trees if it is a red-black tree and if not, which property is violated:




(b) On the following red-black tree, first execute the operation `insert`(8) and afterwards `delete`(5). Draw the resulting tree and document intermediate steps.

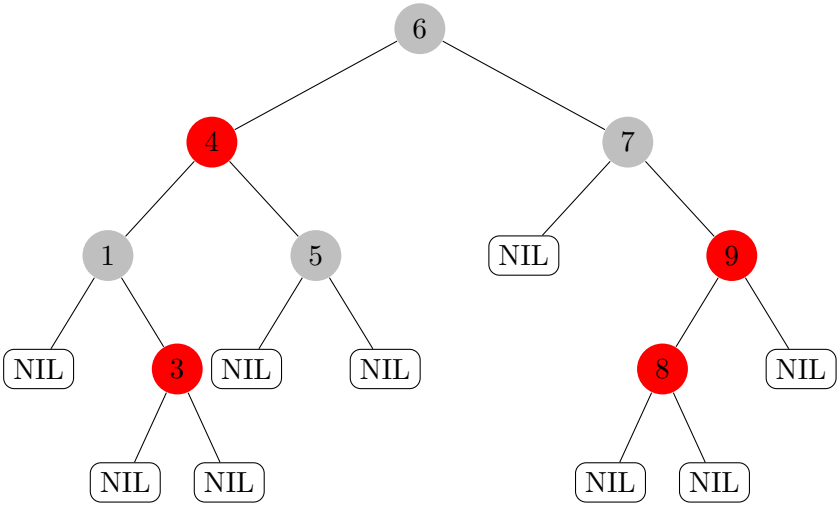



## Sample Solution

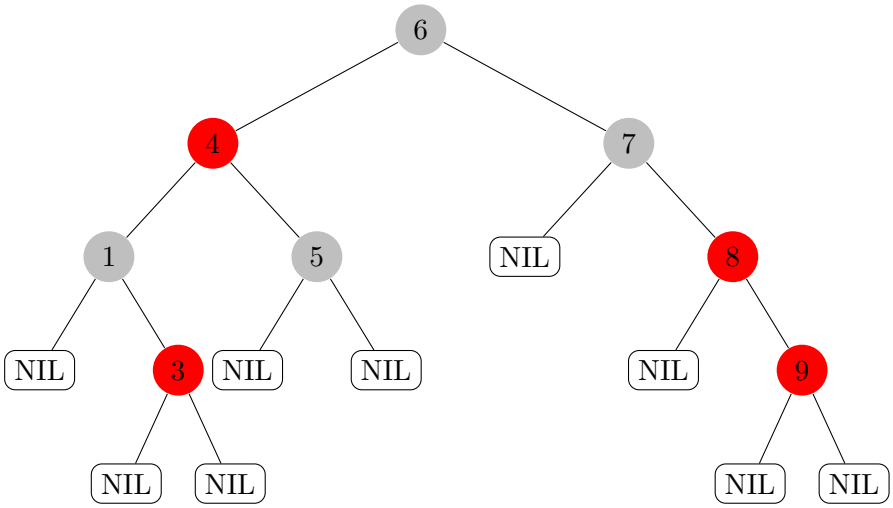
(a) From left to right:

1) Red-black-tree
2) No red-black-tree, because it is no binary search tree (the root's right child has a smaller key).
3) No red-black-tree, because the number of black nodes on a path from the root to a leaf is larger if you go through the left subtree.

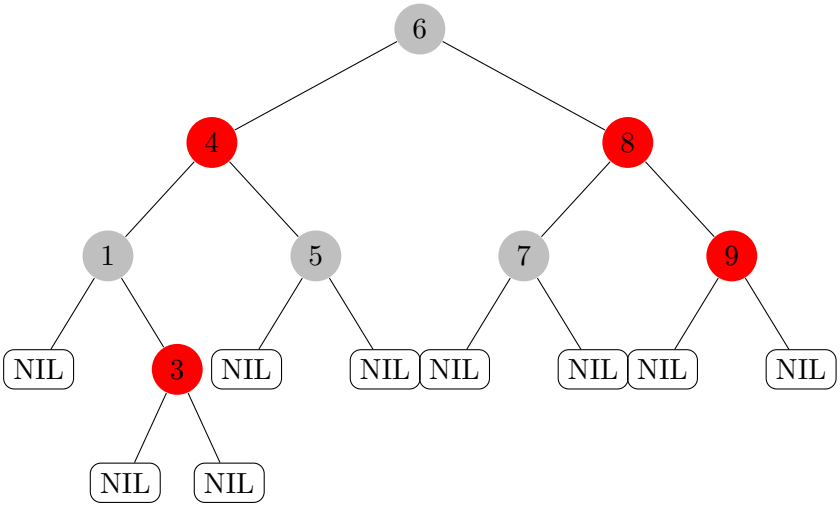
(b) We insert a red node with key 8 according to the rule of inserting into binary search trees.



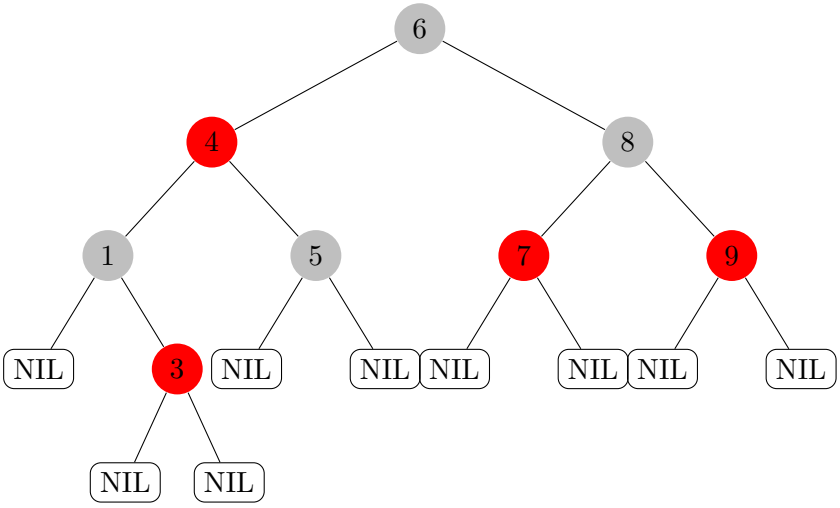

We are in case 1b from the lecture. We do a right-rotate(9,8),



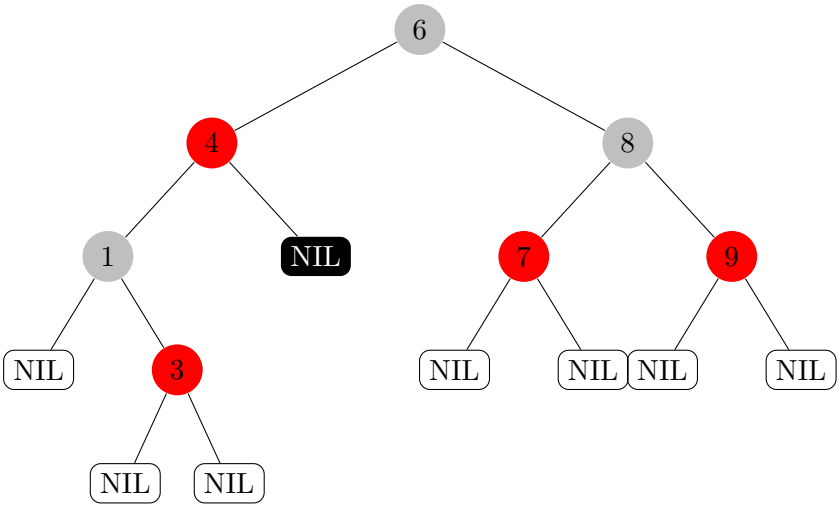

a left-rotate(8,7)



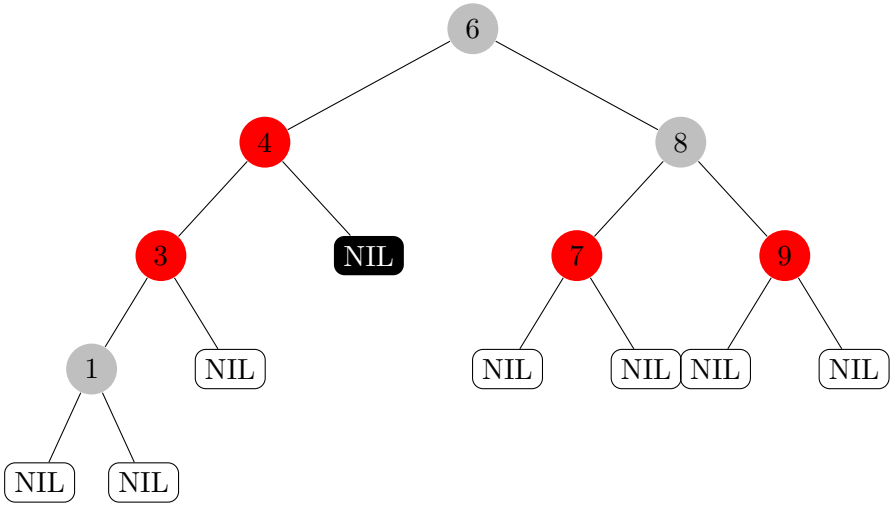

and recolor nodes 7 and 8.



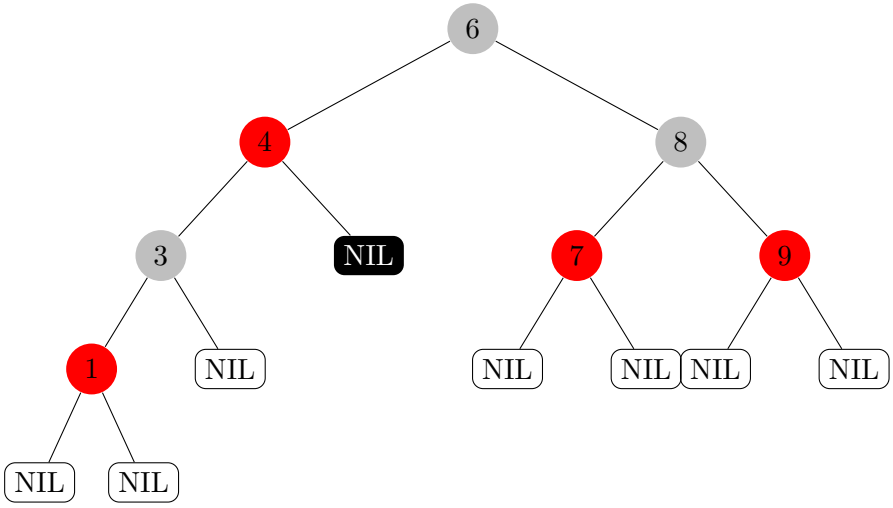

Now we execute `delete`(5). We are in case 2b from the lecture (deleting a black node with two NIL-children). First we remove node 5 from the tree and color the right NIL-child of node 4 double black to correct the black height.



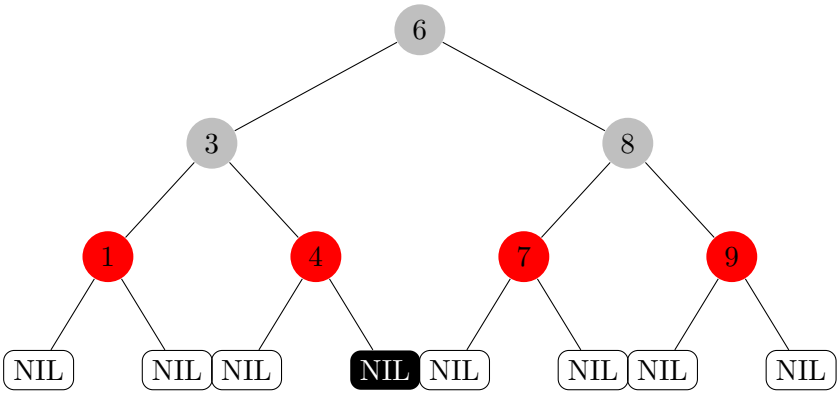

We are in case A.2 from the lecture. We do a left-rotate(3,1)



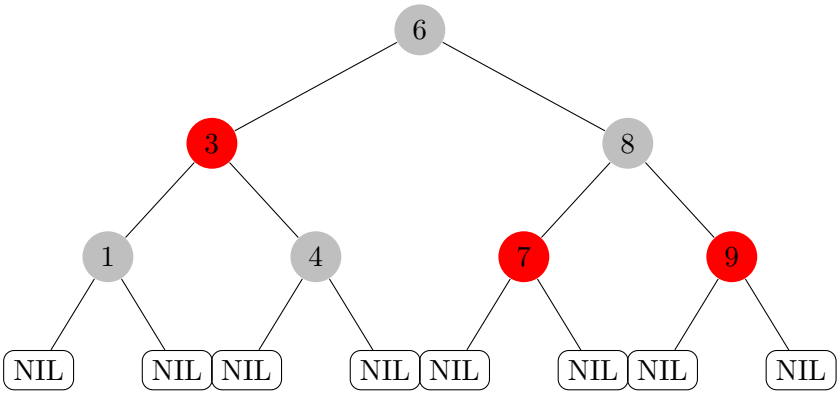

and recolor nodes 1 and 3.




Now we are in case A.1. We do a right-rotate(3,4)




and recolor. Finally, the tree looks like this.




## Exercise 2: AVL-Trees

An AVL-tree is a binary search tree with the additional property that for each node $v$, the depth of its left and its right subtree differ by at most 1.

(a) Show via induction that an AVL-tree of height $d$ is filled completely up to depth $\lfloor \frac{d}{2} \rfloor$.

*A binary tree is filled completely up to depth $d'$ if it contains for all $x \leq d'$ exactly $2^x$ nodes of depth $x$.*

(b) Give a recursion relation that describes the minimum number of nodes of an AVL-tree as a function of $d$.

(c) Show that an AVL-tree with $n$ nodes has depth $\mathcal{O}(\log n)$.

*You can either use part (a) or part (b).*

## Sample Solution

(a) **Induction start:** Each non-empty tree has a root and is hence completely filled up to depth 0. Hence the statement is true for $d = 0$ and $d = 1$ (as $\lfloor d/2 \rfloor = 0$ for $d = 0$ and $d = 1$).

**Induction step:** Assume the statement holds for all AVL-trees up to depth $d$. We show that it also holds for AVL-trees of depth $d + 1$.

Let $T$ be an AVL-tree of depth $d + 1$ with $r$ as root and $T_\ell$ and $T_r$ as left and right subtree. One of these subtrees must have depth $d$ (lets say $T_\ell$). As $T$ is an AVL-tree, it follows that $T_r$ has depth at least $d - 1$. By the induction hypothesis, $T_\ell$ is completely filled up to depth $\lfloor d/2 \rfloor$ and $T_r$ is completely filled up to depth $\lfloor \frac{d-1}{2} \rfloor$. So both subtrees are completely filled up to depth $\lfloor \frac{d-1}{2} \rfloor = \lfloor \frac{d+1}{2} - 1 \rfloor = \lfloor \frac{d+1}{2} \rfloor - 1$ and hence $T$ is filled completely up to depth $\lfloor \frac{d+1}{2} \rfloor$.

(b) Let $n_d$ be the minimum number of nodes in an AVL-tree of depth $d$. As every tree of depth $d$ has at least $d + 1$ nodes (as it contains a path of length $d$), we obtain as base cases $n_0 = 1$ and $n_1 = 2$. Now let $d \geq 2$. An AVL-tree $T$ of depth $d$ consists of a root $r$, a left subtree $T_\ell$ and a right subtree $T_r$. One of them, lets say $T_\ell$, has depth $d - 1$ and hence at least $n_{d-1}$ nodes. As $T$ is an AVL-tree, it follows that $T_r$ has depth at least $d - 2$ and hence at least $n_{d-2}$ nodes. Hence $T$ has at least $n_d = n_{d-1} + n_{d-2} + 1$ nodes.

(c) **Using (a)**: And AVL-tree of depth $d$ is filled completely up to depth $\lfloor \frac{d}{2} \rfloor$, so $T$ has $n \geq 2^{\lfloor \frac{d}{2} \rfloor}$ nodes. We obtain

$$
\begin{aligned}
& 2^{\lfloor \frac{d}{2} \rfloor} \leq n \\
\iff \quad & \lfloor \frac{d}{2} \rfloor \leq \log(n) \\
\implies \quad & \frac{d}{2} - \frac{1}{2} \leq \lfloor \frac{d}{2} \rfloor \leq \log(n) \\
\implies \quad & d \leq 2 \log n + 1 \\
\implies \quad & d \in \mathcal{O}(\log(n)).
\end{aligned}
$$

**Using (b)**: Similar to the Fibonacci-series we have $n_d = n_{d-1} + n_{d-2} + 1 = 2n_{d-2} + n_{d-3} + 2 \geq 2n_{d-2}$. This means that increasing the depth by 2 doubles the number of nodes, so the number of nodes grows exponentially in the depth, or the depth grows logarithmically in the number of nodes. More formally, we have $n_d \geq 2n_{d-2} \geq 2^2 n_{d-4} \geq \cdots \geq 2^{\lfloor d/2 \rfloor} n_{d-2\lfloor d/2 \rfloor} \geq 2^{\lfloor d/2 \rfloor} n_0 = 2^{\lfloor d/2 \rfloor}$. The rest follows as above.