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Algorithms and Datastructures
Winter Term 2024
Exercise Sheet 5

Due: Wednesday, May 29rd, 2pm

Exercise 1: Bad Hash Functions *(10 Points)*

Let m be the size of a hash table and $M \gg m$ the largest possible key of the elements we want to store in the table. The following "hash functions" are poorly chosen. Explain for each function why it is not a suitable hash function.

(a) $h : x \mapsto \lfloor \frac{x}{m} \rfloor \bmod m$ *(1,5 Points)*

(b) $h : x \mapsto (2x+1) \bmod m$ (m even). *(1,5 Points)*

(c) $h : x \mapsto (x \bmod m) + \lfloor \frac{m}{x+1} \rfloor$ *(1,5 Points)*

(d) For each calculation of the hash value of x one chooses for $h(x)$ a uniform random number from $\{0, \ldots, m-1\}$ *(1,5 Points)*

(e) $h : x \mapsto \lfloor \frac{M}{x \cdot p \bmod M} \rfloor \bmod m$, where p is prime and $\frac{M}{2} < p < M$ *(2 Points)*

(f) For a set of "good" hash functions $h_1, \ldots, h_\ell$ with $\ell \in \Theta(\log m)$, we first compute $h_1(x)$, then $h_2(h_1(x))$ etc. until $h_\ell(h_{\ell-1}(\ldots h_1(x))\ldots)$. That is, the function is $h : k \mapsto h_\ell(h_{\ell-1}(\ldots h_1(x))\ldots)$ *(2 Points)*

Exercise 2: (No) Families of Universal Hash Functions *(10 Points)*

(a) Let $\mathcal{S} = \{0, \ldots, M-1\}$ and $\mathcal{H}_1 := \{h : x \mapsto a \cdot x^2 \bmod m \mid a \in \mathcal{S}\}$. Show that H_1 is not c-univeral for *constant* $c \geq 1$ (that is c is fixed and must not depend on m). *(4 Points)*

(b) Let m be a *prime* and let $k = \lfloor \log_m M \rfloor$. We consider the keys $x \in \mathcal{S}$ in base m presentation, i.e., $x = \sum_{i=0}^{k} x_i m^i$. Consider the set of functions from the lecture (week 5, slide 15)

$$\mathcal{H}_2 := \Big\{ h : x \mapsto \sum_{i=0}^{\mathbf{k}} a_i x_i \mod m \;\Big|\; a_i \in \{0, \ldots, m-1\} \Big\}.$$

Show that $\mathcal{H}_2$ is 1-universal. *(6 Points)*

Hint: Two keys $x \neq y$ have to differ at some digit $x_j \neq y_j$ in their base m presentation.
Remark: Since m is prime, for each element $a \in \{1, ..., m-1\}$ there exists an inverse element $b \in \{1, ..., m-1\}$ of a modulo m i.e., $a \cdot b \equiv 1 \mod m$.