University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
S. Faour, A. Mályusz

Theory of Distributed Systems
Exercise Sheet 7

Due: Wednesday, 19th of June 2024, 12:00 noon

Exercise 1: Matching *(5 Points)*

A *matching* of a graph $G = (V, E)$ is a subset of edges $M \subseteq E$ such that no two edges in M are adjacent. A matching is maximal if no edge can be added without violating this property.

Give an algorithm that computes a maximal matching in $O(\log n)$ rounds w.h.p. in the synchronous message passing model. That is, after the algorithm terminates each node needs to know which of its adjacent edges are part of the maximal matching.

Exercise 2: Dominating Set *(8 Points)*

A *dominating set* of a graph $G = (V, E)$ is a subset of the nodes $D \subseteq V$ such that each node is in D or adjacent to a node in D. A minimum dominating set is a dominating set containing the least possible number of nodes. $G = (V, E)$ has *neighborhood independence* β if for every node $v \in V$ the largest independent set of the neighborhood $N(v) := \{u \in V \mid \{v, u\} \in E\}$ of v is of size at most β.

a) Show that for an MIS M and a minimum dominating set D of a graph it holds $|D| \leq |M|$.

b) Give a class of graphs each containing an independent set I and a dominating set D with $\frac{|I|}{|D|} = O(n)$.

c) Show that for graphs with neighborhood independence $\beta \geq 1$, a β-approximation to a minimum dominating set (that is a dominating set which is at most β times larger than a minimum dominating set) can be found in time $O(\log n)$ w.h.p. in the synchronous message passing model.

d) A unit disc graph is a graph (V, E) with $V \subset \mathbb{R}^2$ and $E = \{\{u, v\} \mid \|u - v\|_2 \leq 1\}$. Show that one can compute a 5-approximation to a minimum dominating set in disc graphs in time $O(\log n)$ w.h.p. in the synchronous message passing model.

Exercise 3: Coloring *(7 Points)*

Assume we have $C = \alpha(\Delta + 1) \in \mathbb{N}$ colors for some $\alpha \geq 1$. Consider the following algorithm in the synchronous message passing model to color the graph with C colors. Each node v repeats the following steps (corresponding to a phase) until it has a color:

- Let N_v be the set of yet uncolored neighbors of v and let C_v be the set of colors that v's neighbors already chose (initially N_v are all of v's neighbors and $C_v = \emptyset$).
- Node v picks a random number $r_c(v) \in [0, 1]$ for every remaining color $c \in \{1, \ldots, C\} \setminus C_v$ and informs its neighbors about those numbers.
- If $r_c(v) < r_c(u)$ for some $c \in \{1, \ldots, C\} \setminus C_v$ and every $u \in N_v$, then v colors itself with c, informs its neighbors and terminates (if this holds for several c, v chooses one of those arbitrarily).

(a) Show that the probability that a node obtains a color in a given phase is at least $1 - e^{-\alpha}$.

(b) Show that the algorithm terminates after $\mathcal{O}\big(1 + \frac{\log n}{\alpha}\big)$ rounds in expectation.