Randomized Coloring & MIS

Dennis Olivetti

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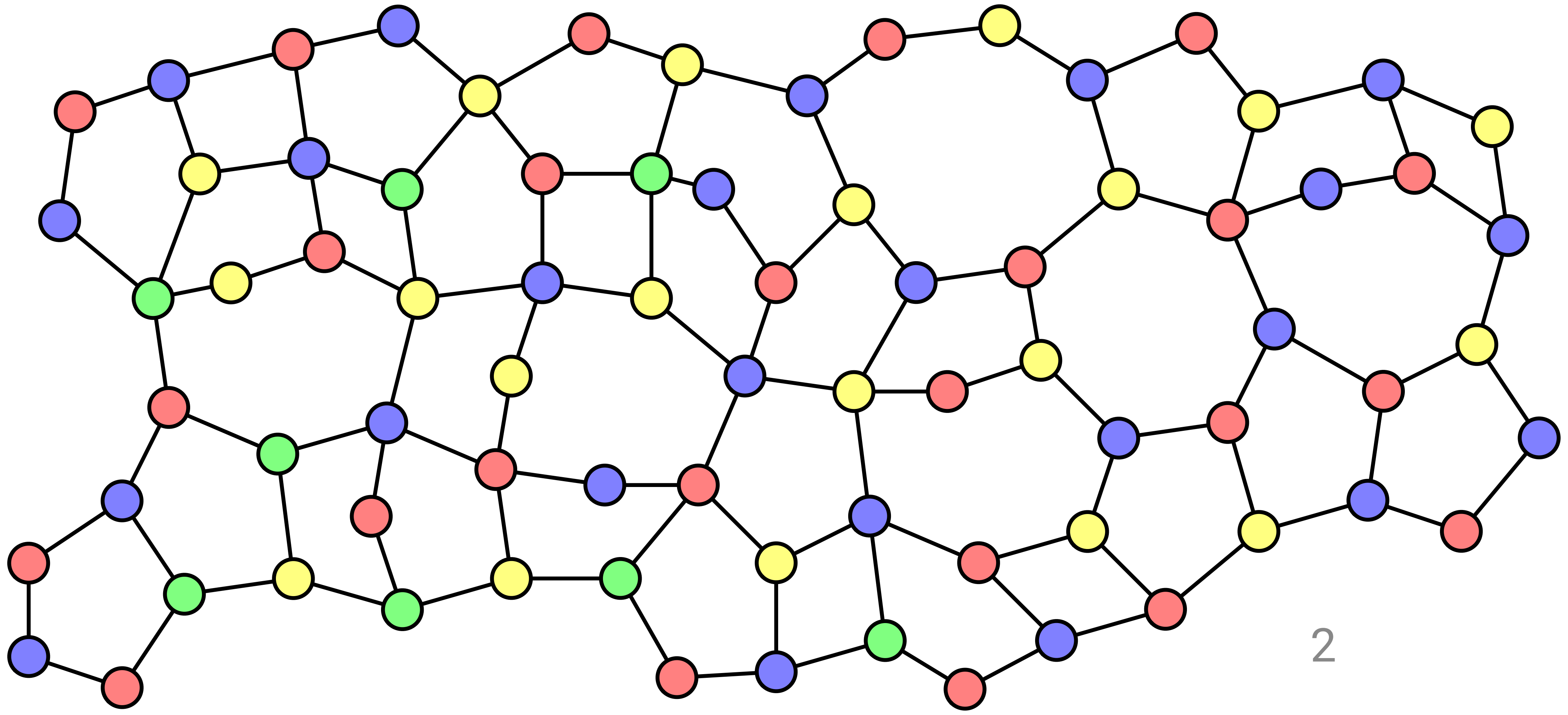
Distributed Coloring Problem

(Δ+1)-Vertex Coloring



Distributed Coloring Problem

Objective: properly color the nodes with **≤ Δ + 1** colors

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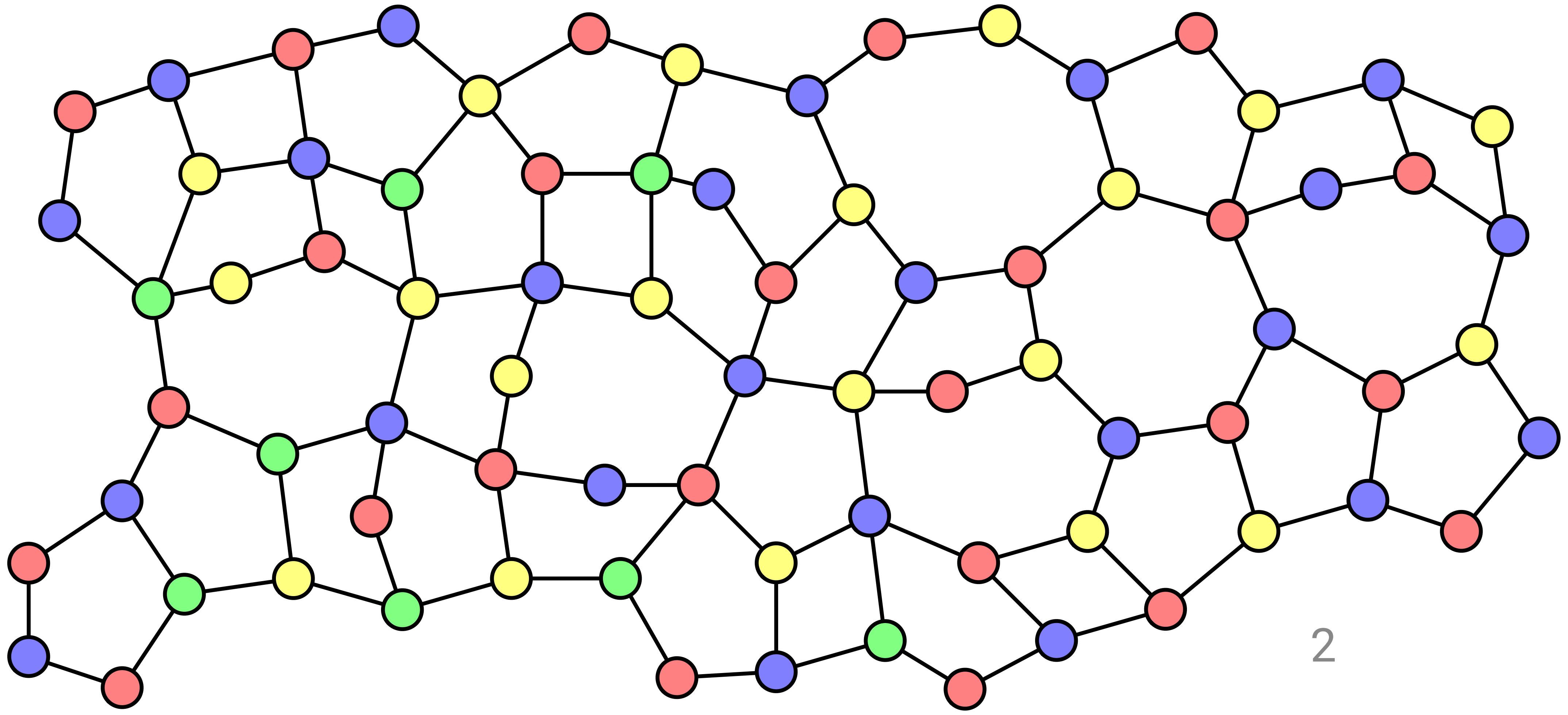

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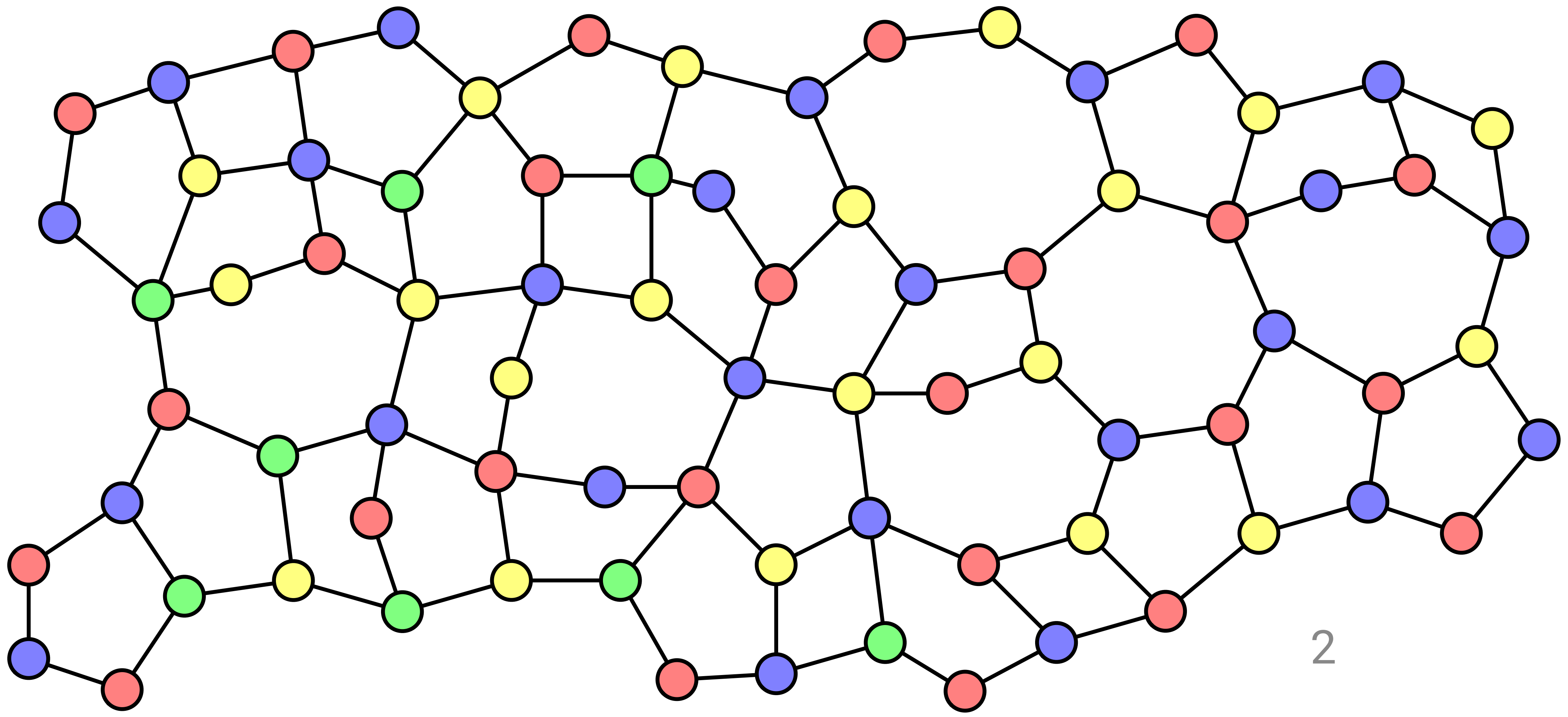

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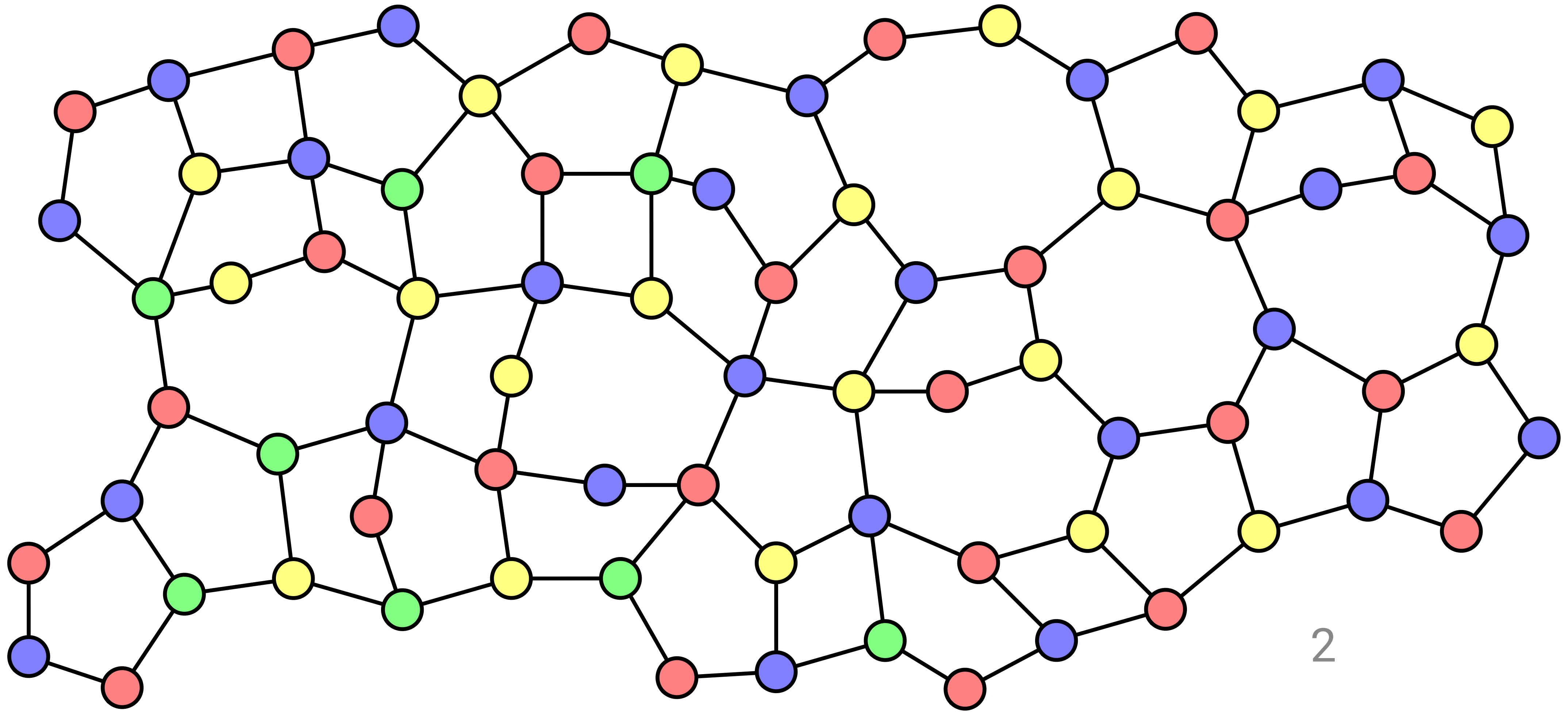

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- Δ + 1: what a simple sequential **greedy** algorithm achieves

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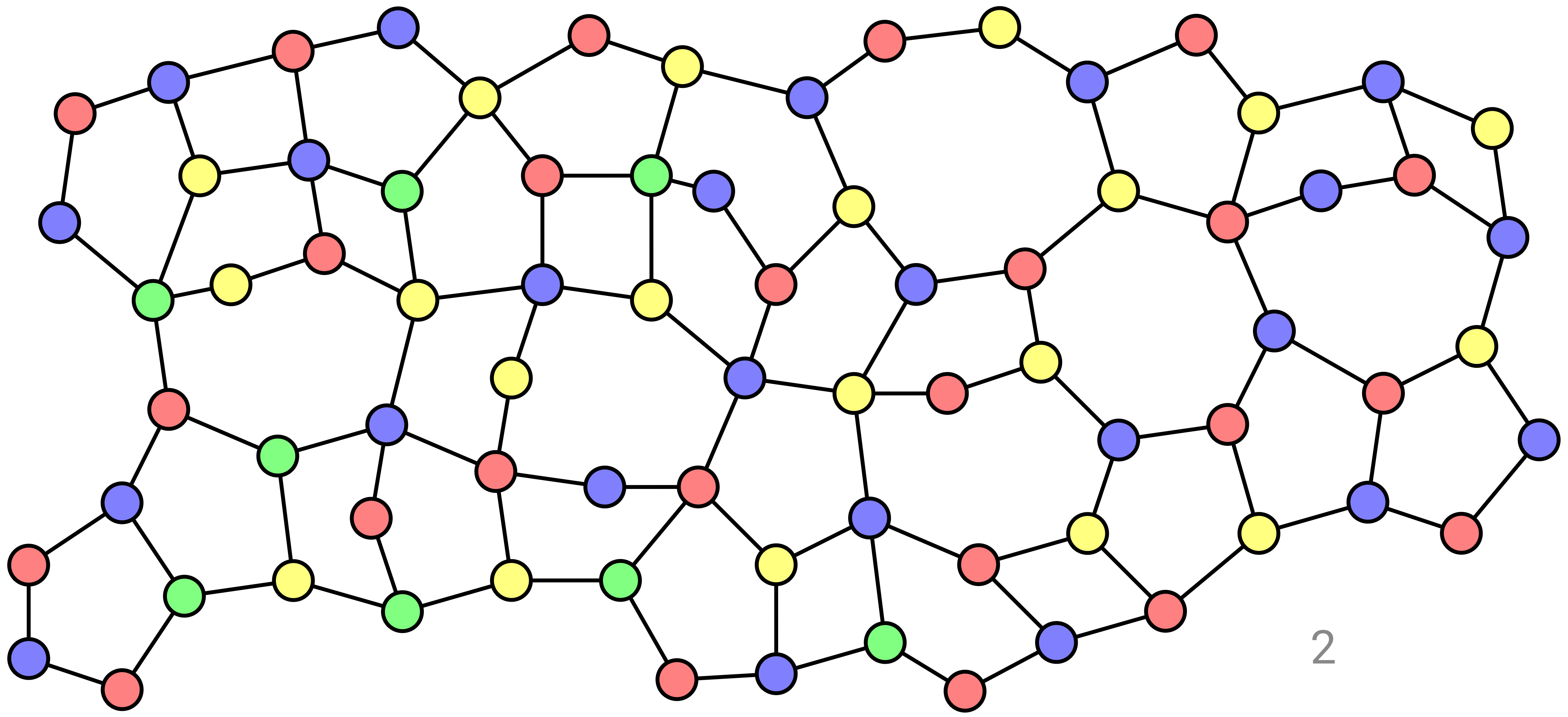
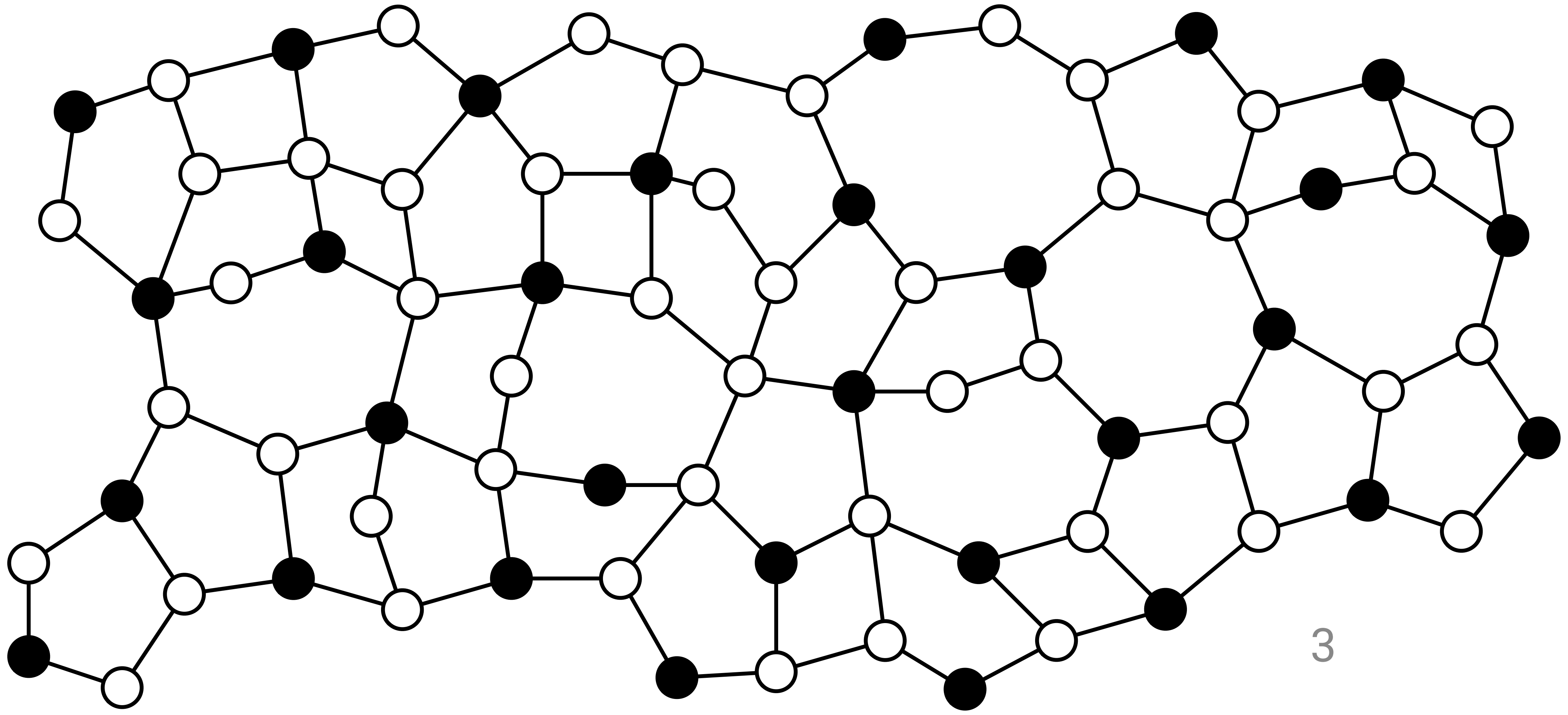


Distributed MIS Problem

Maximal Independent Set



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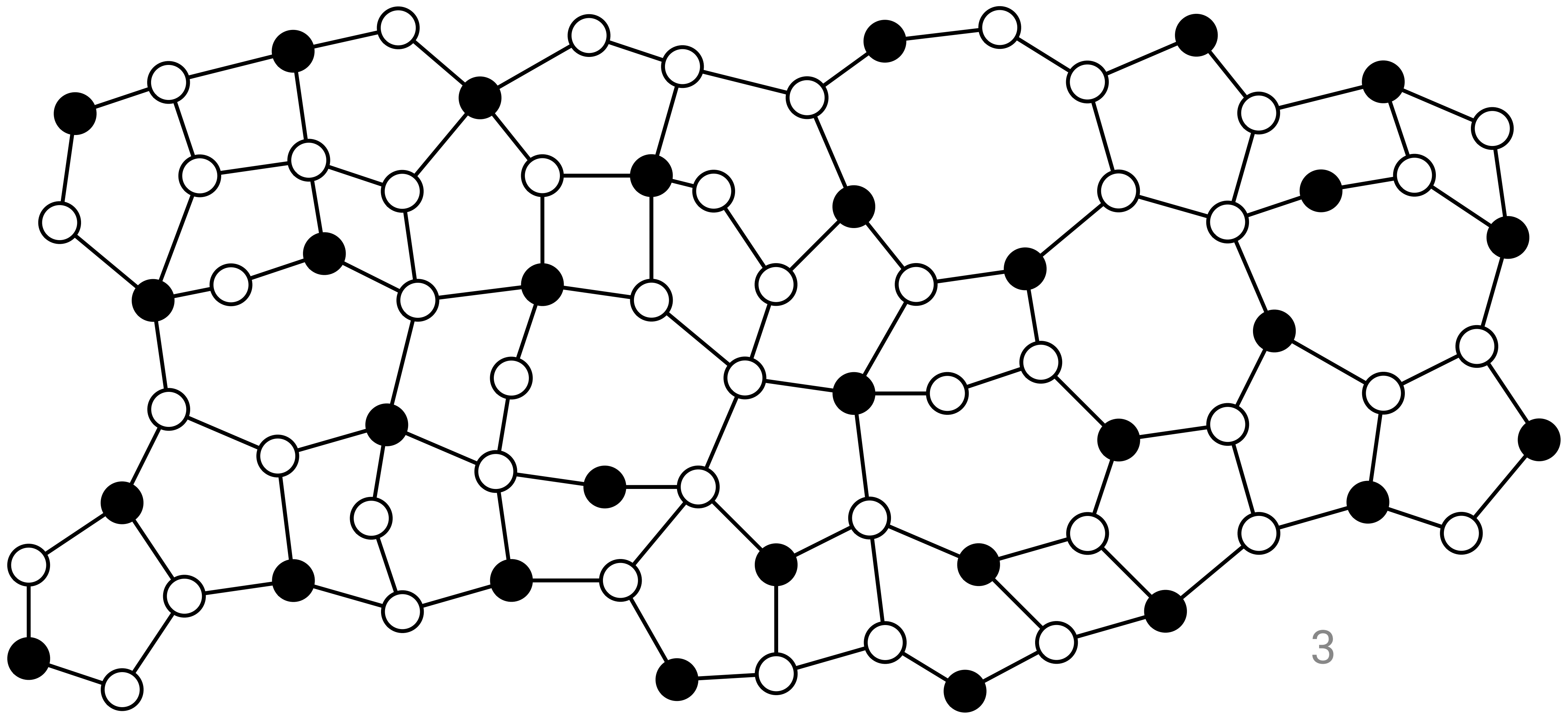

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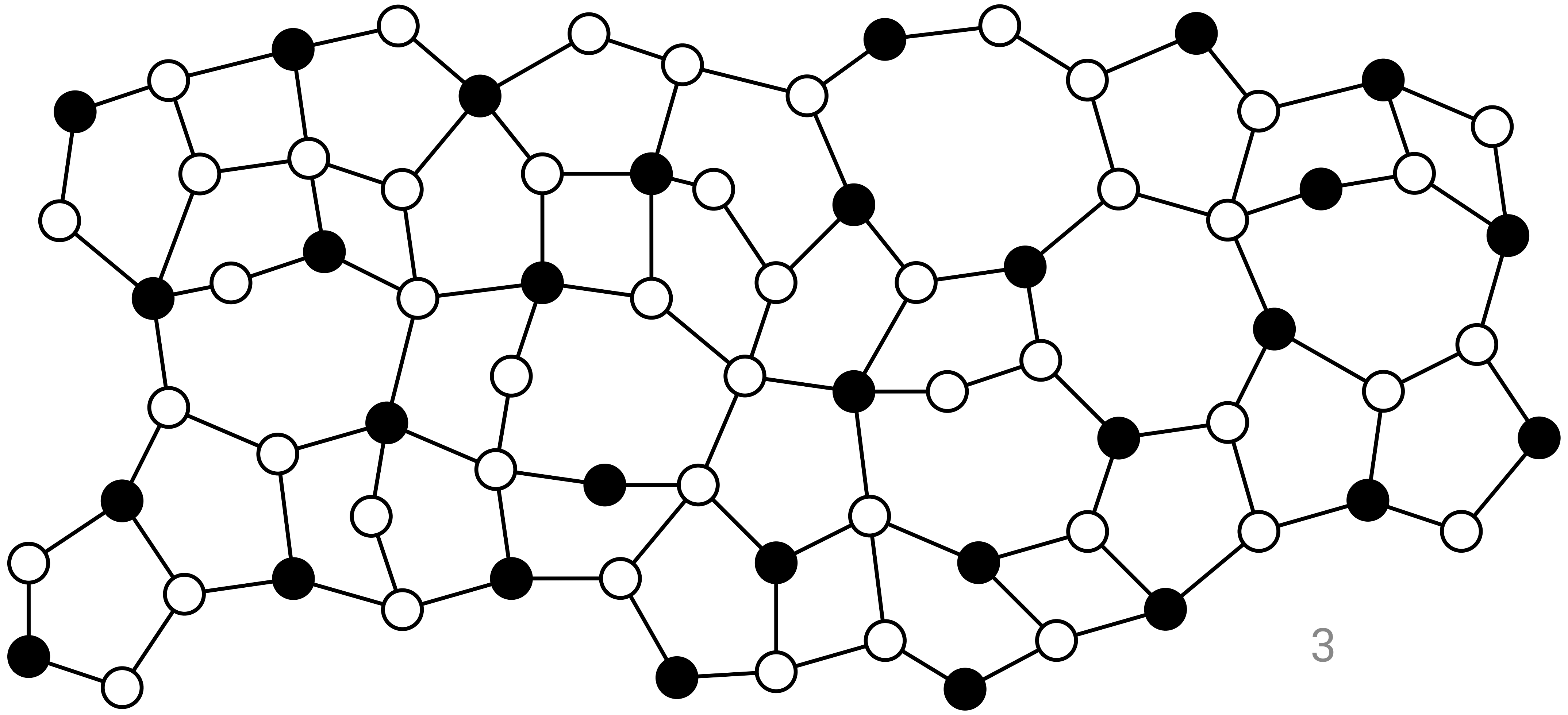

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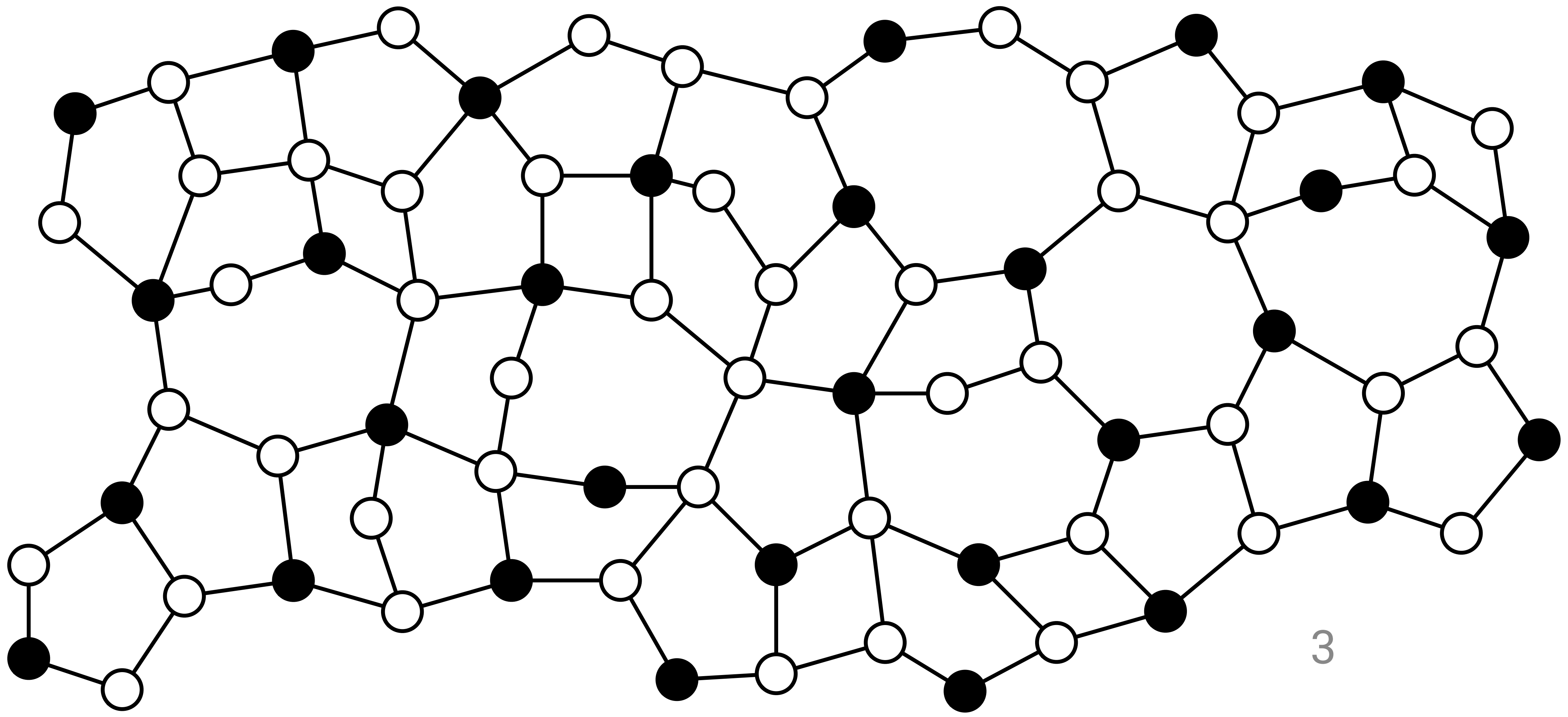

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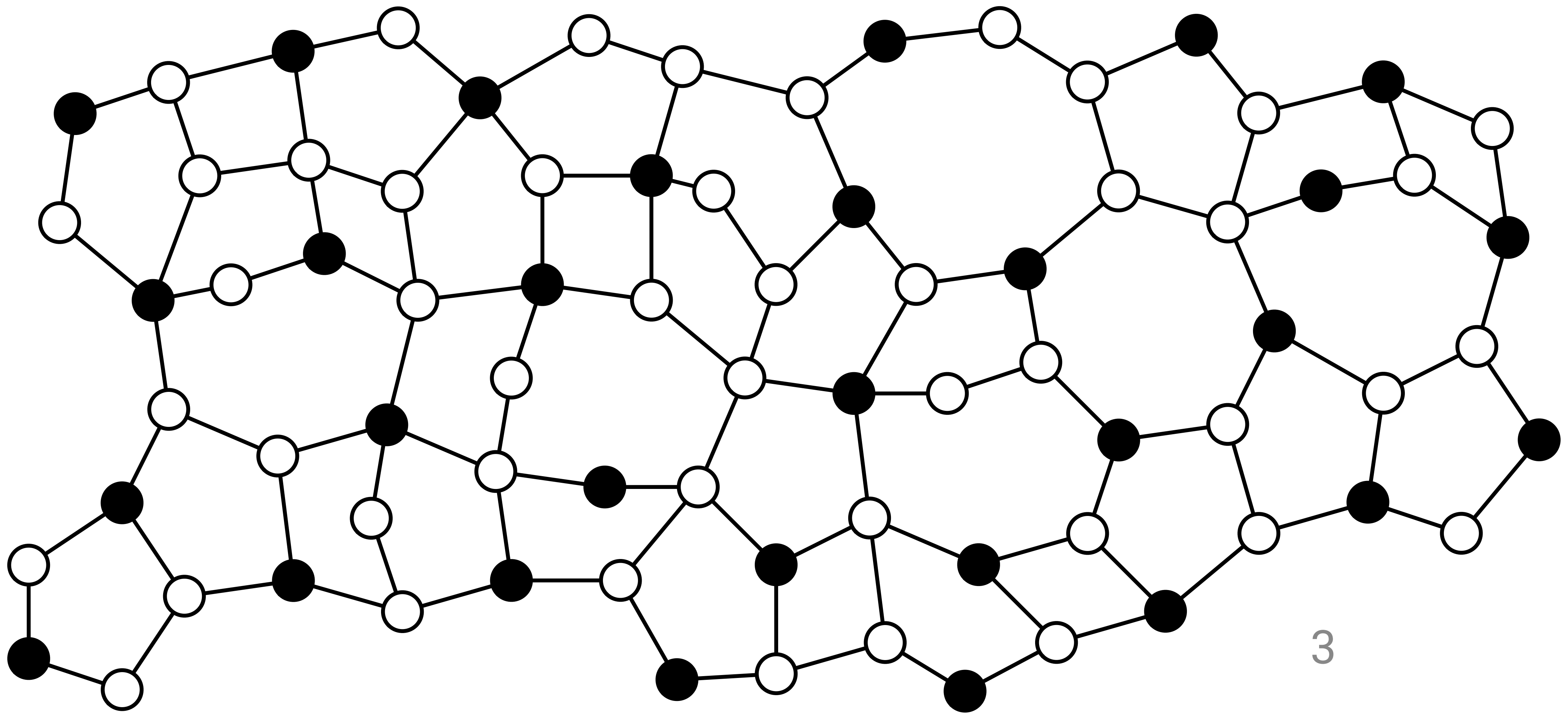

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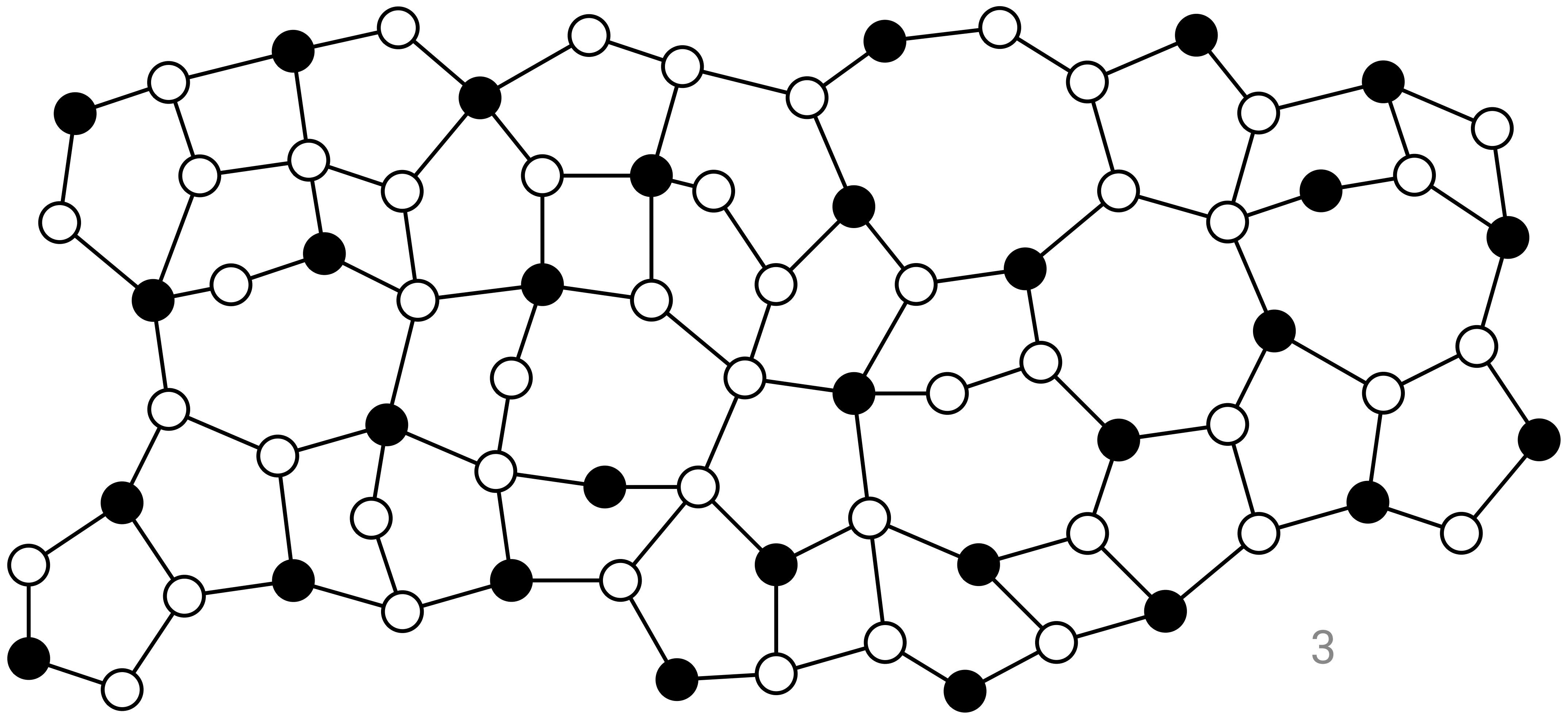

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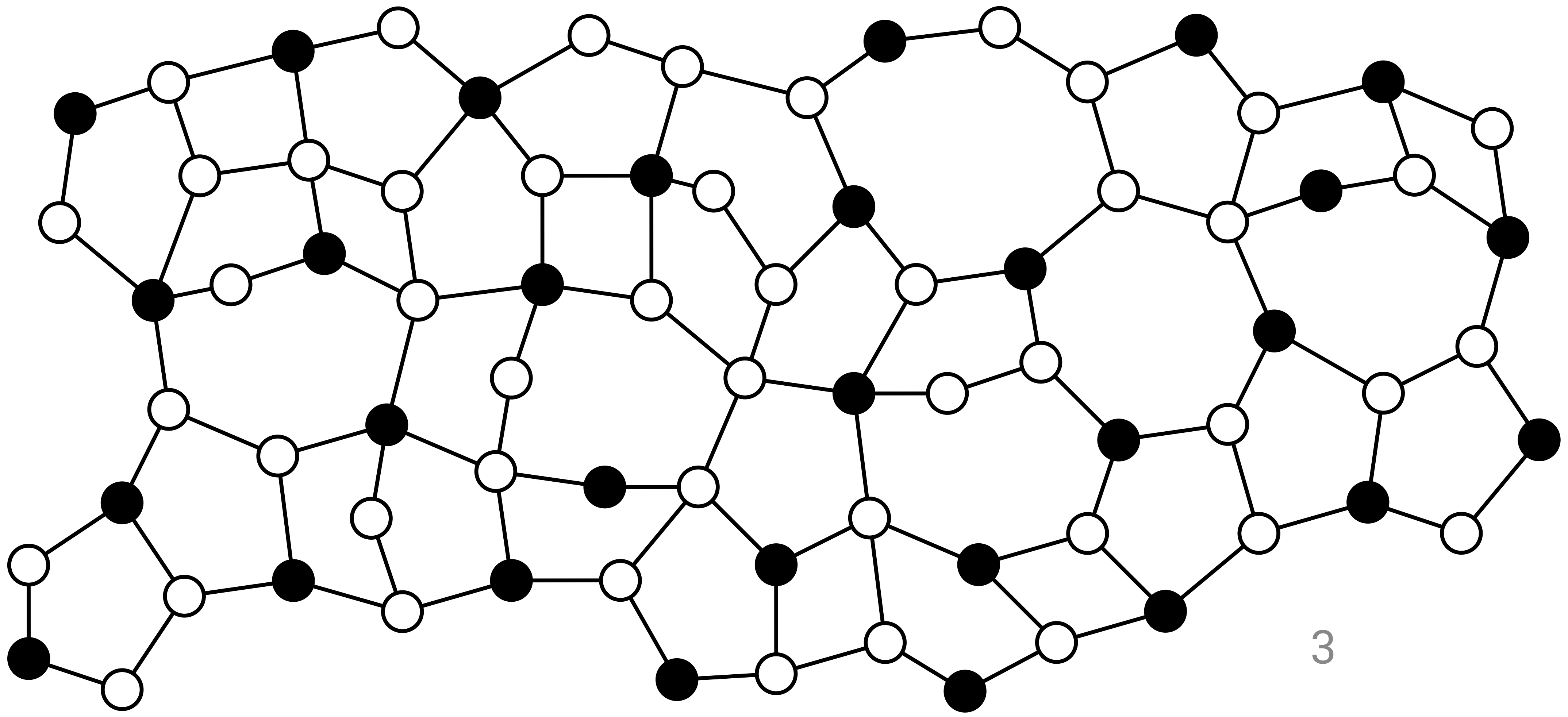

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- **Today**
  - **Randomized algorithms** for $(\Delta+1)$-coloring and MIS: $O(\log n)$ time in general graphs!

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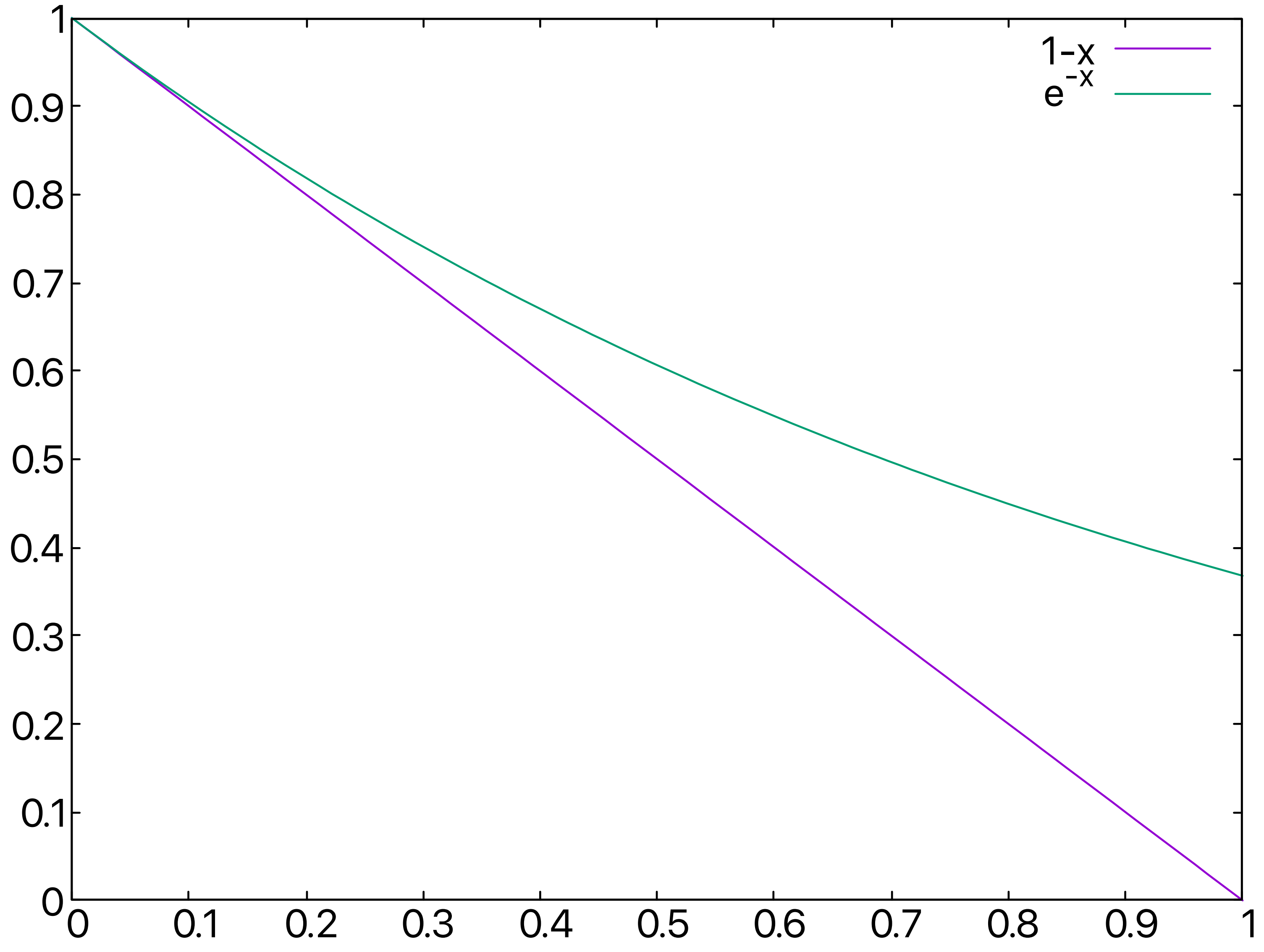

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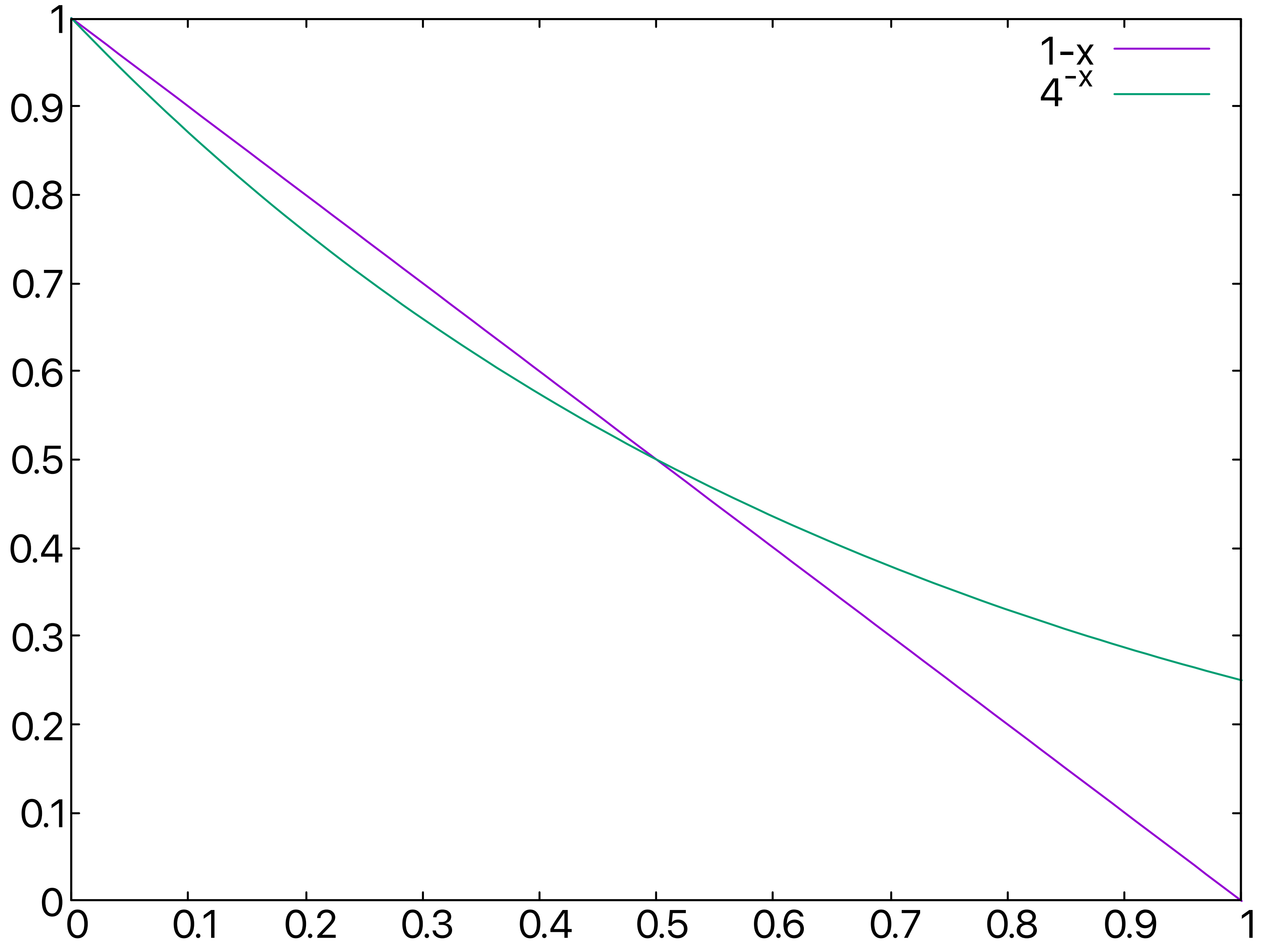

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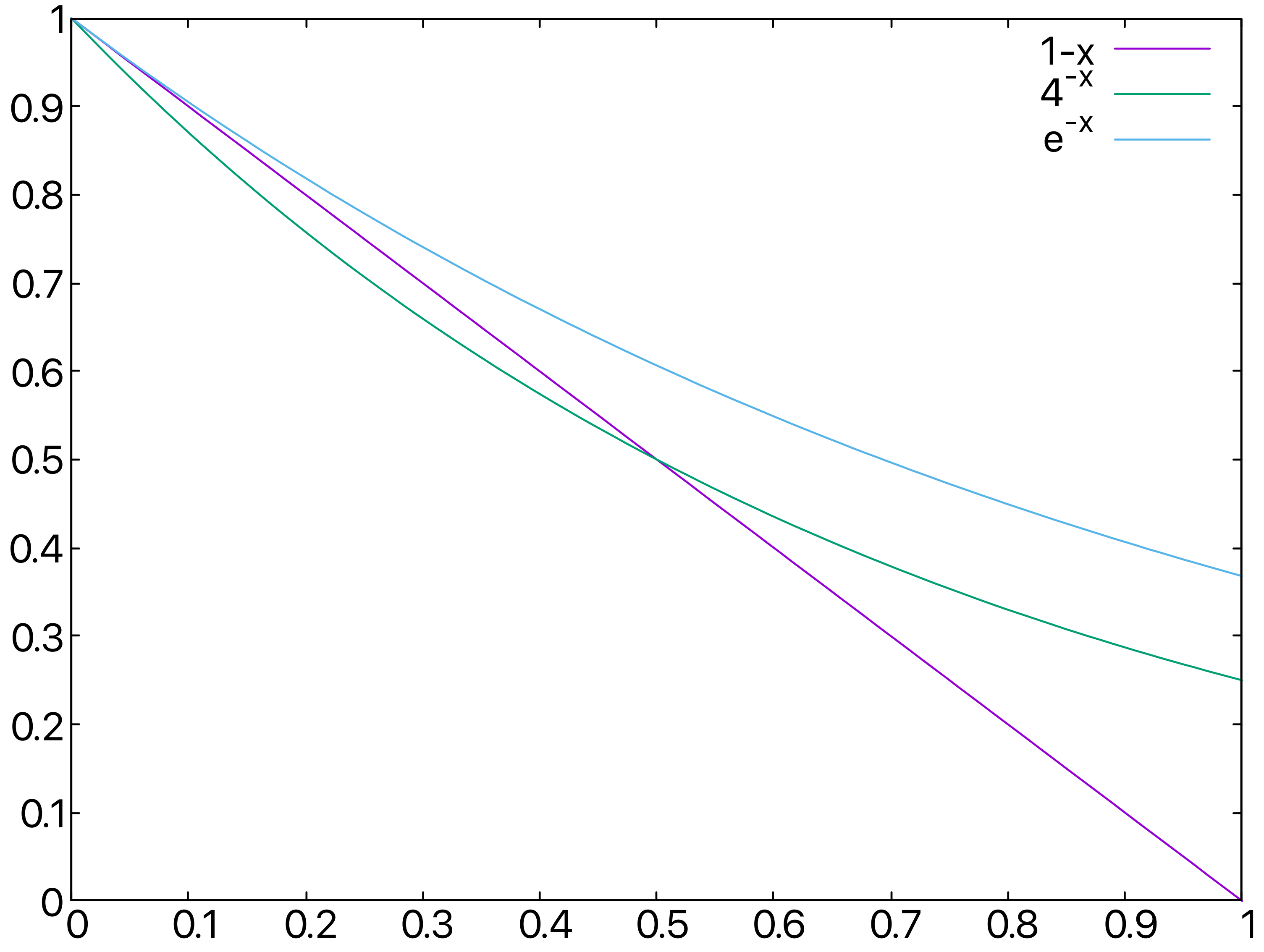

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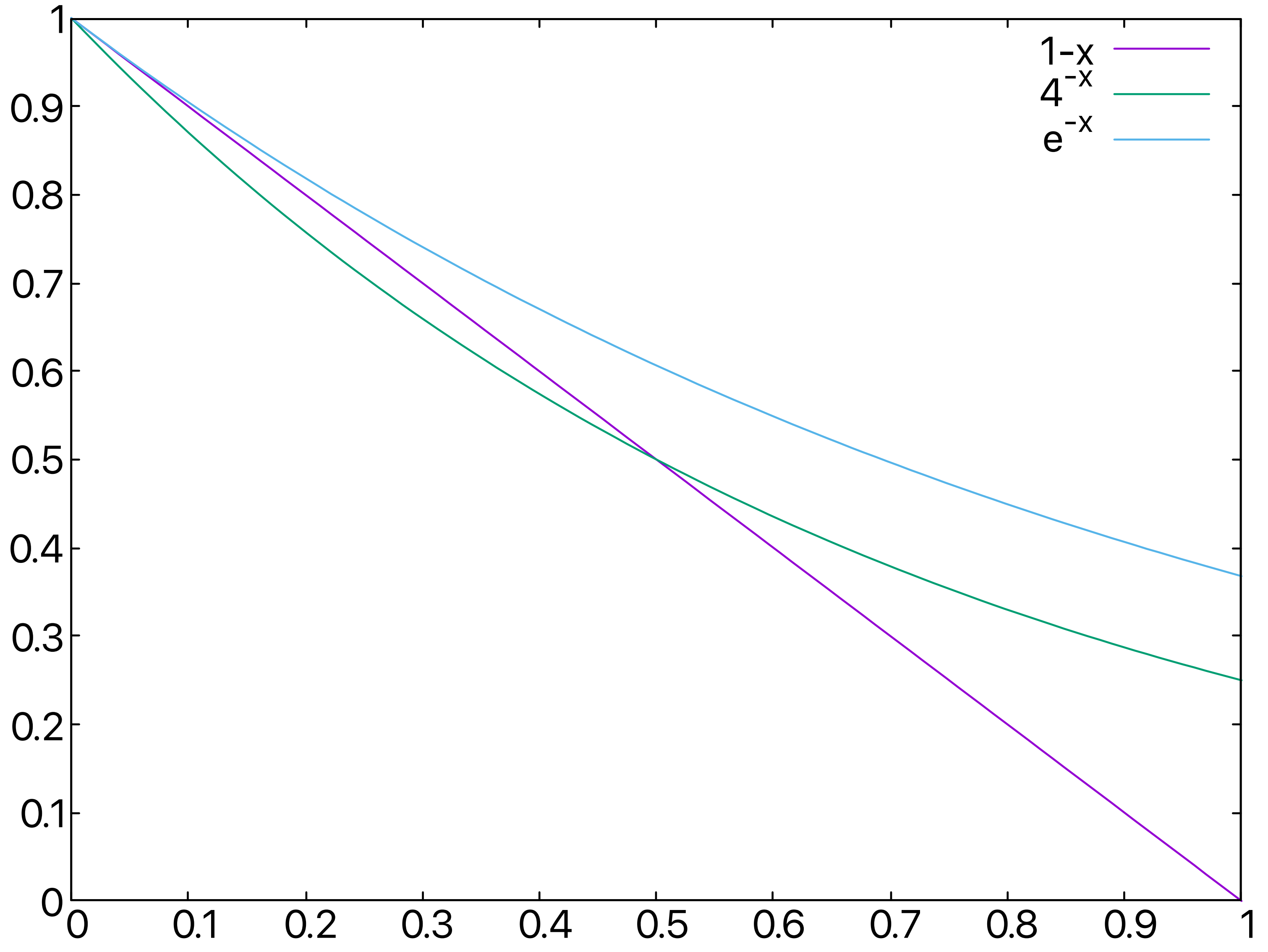
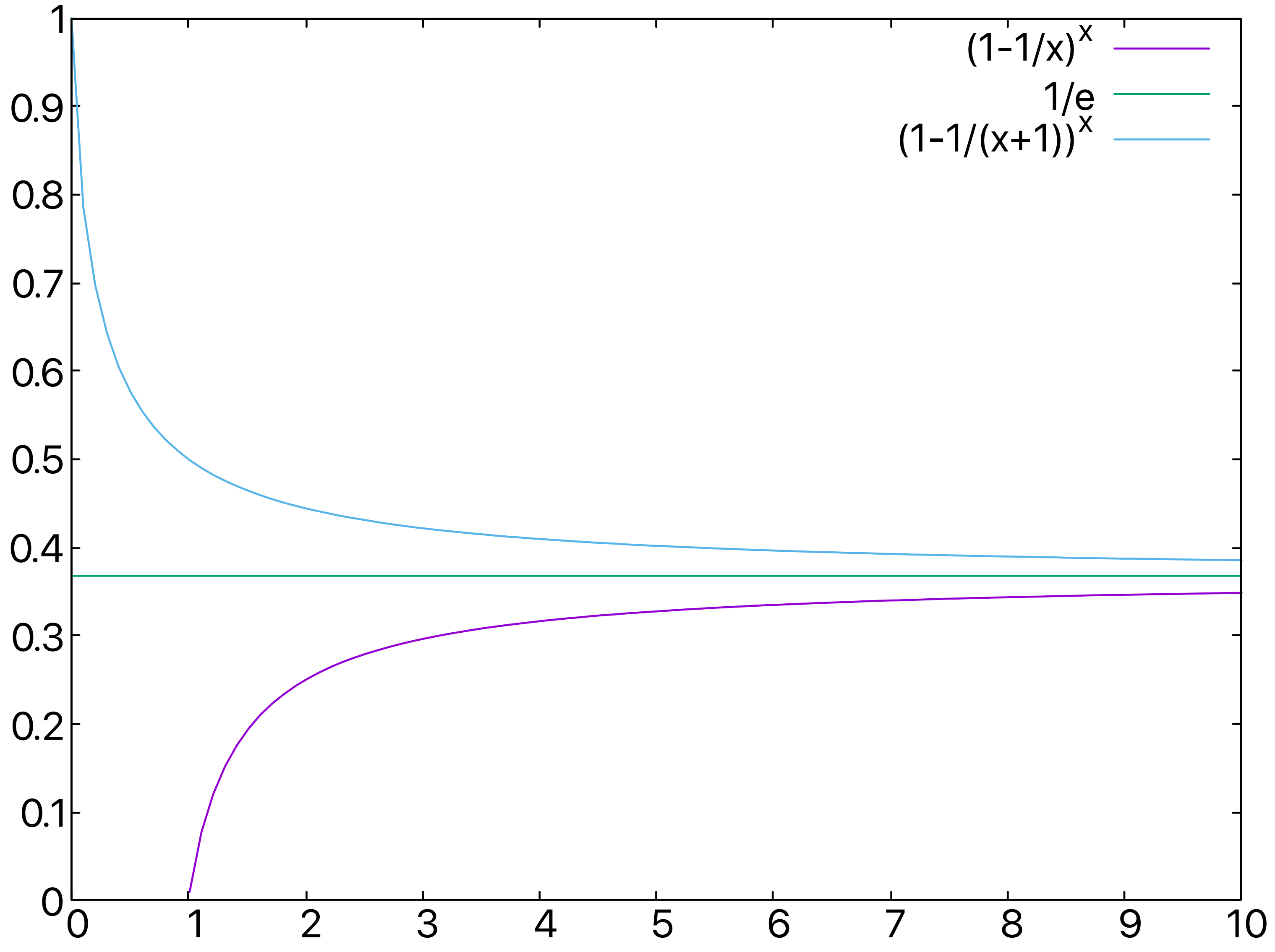
    - $4^{-x} \leq 1 - x \leq e^{-x}$ for all $x \in [0, 1/2]$
- $\lim_{x\to\infty} (1 - 1/x)^x = 1/e$
    - $(1 - 1/x)^x < 1/e$ for all $x \geq 1$
    - $(1 - 1/(x+1))^x > 1/e$ for all $x > 0$





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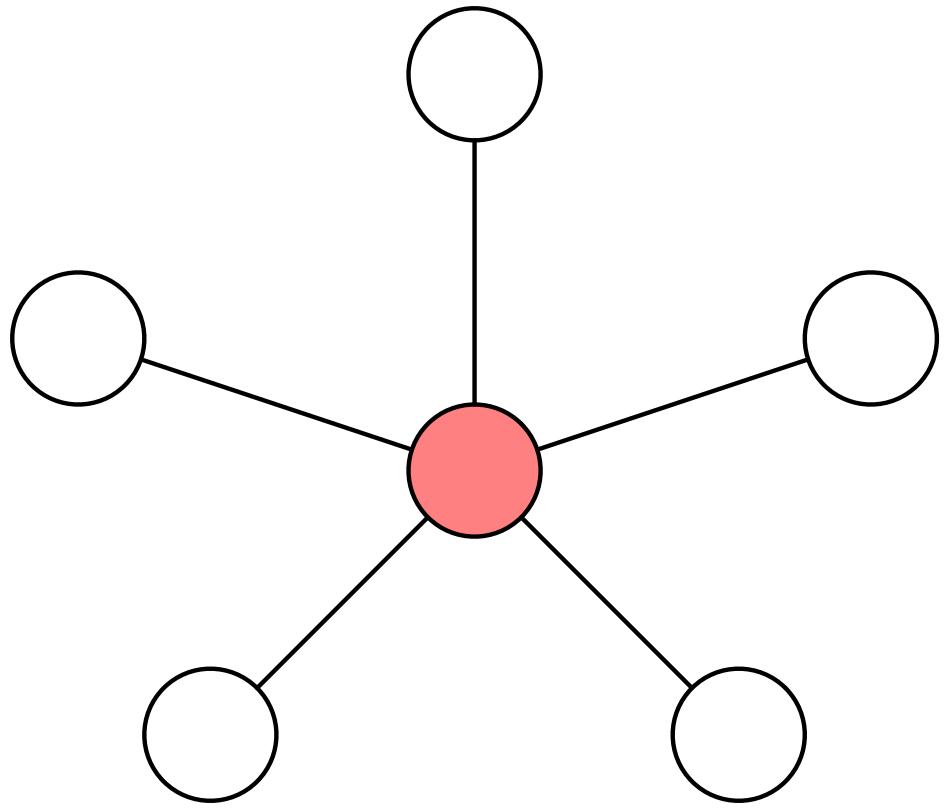

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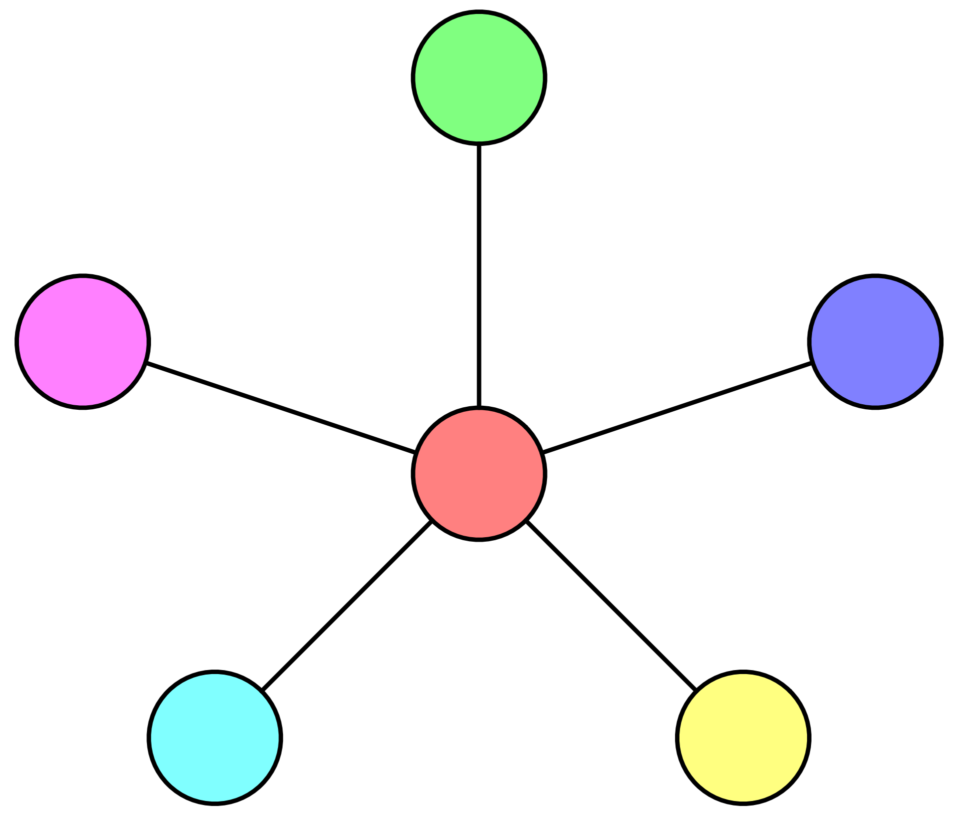

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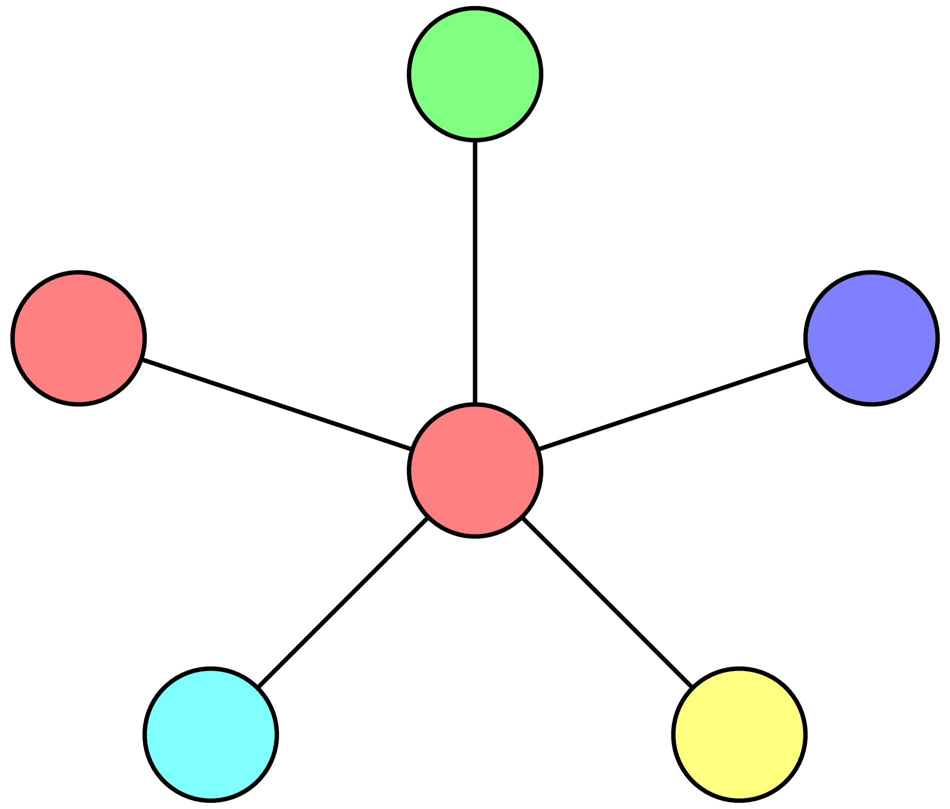

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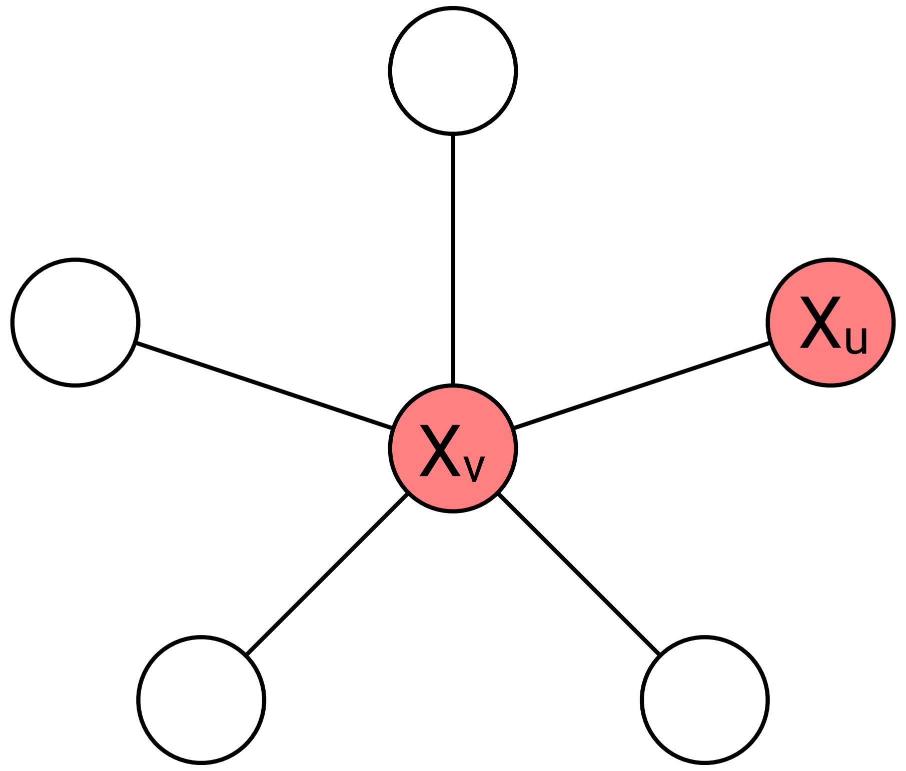

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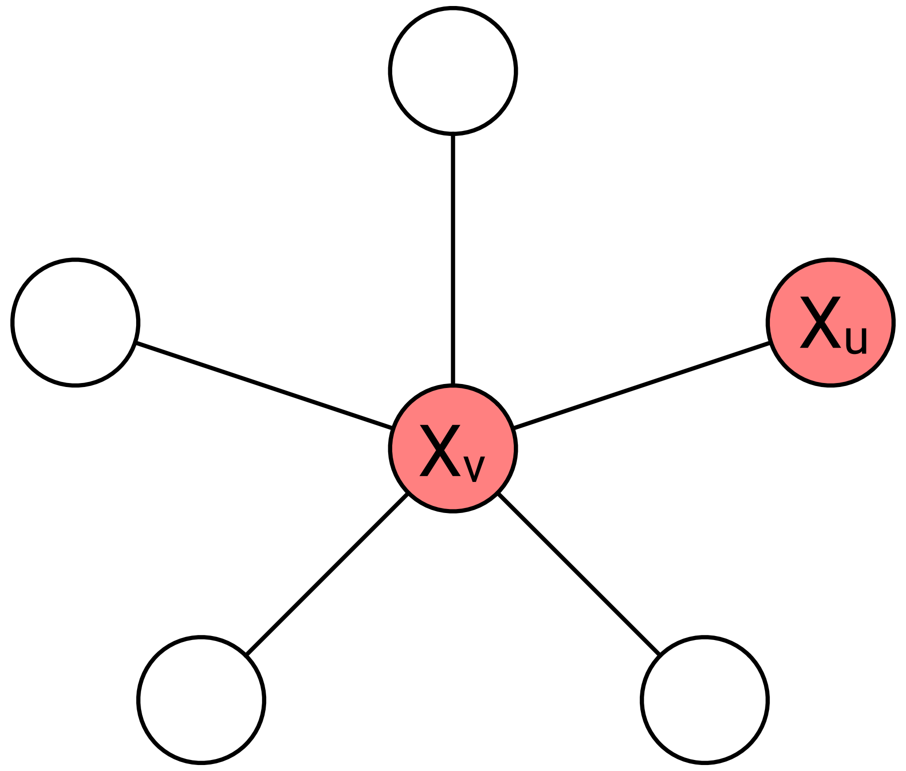

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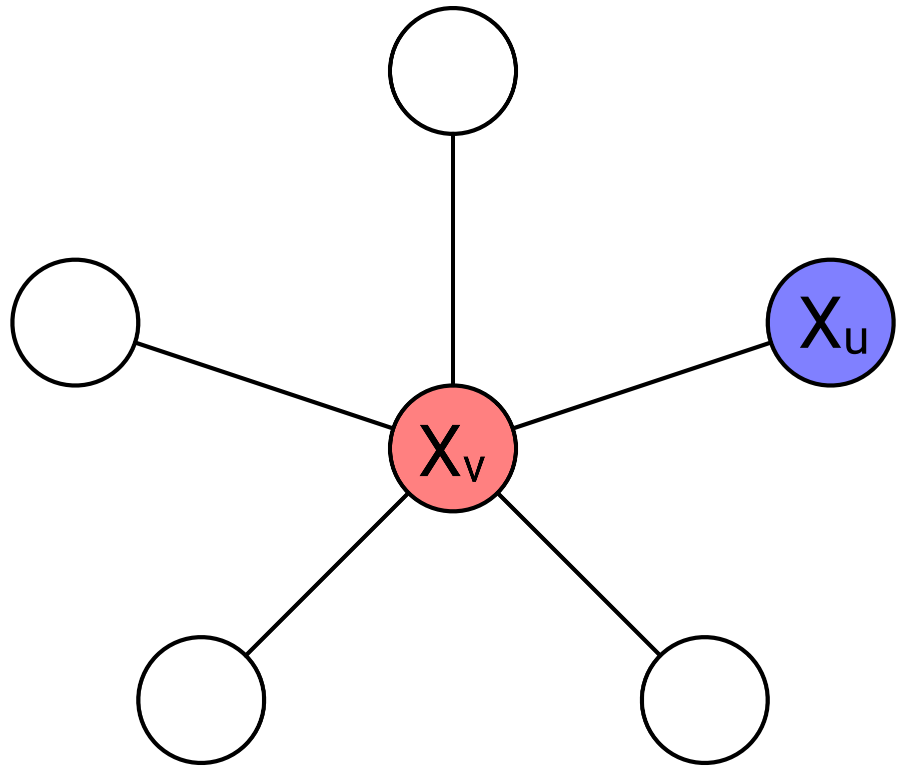

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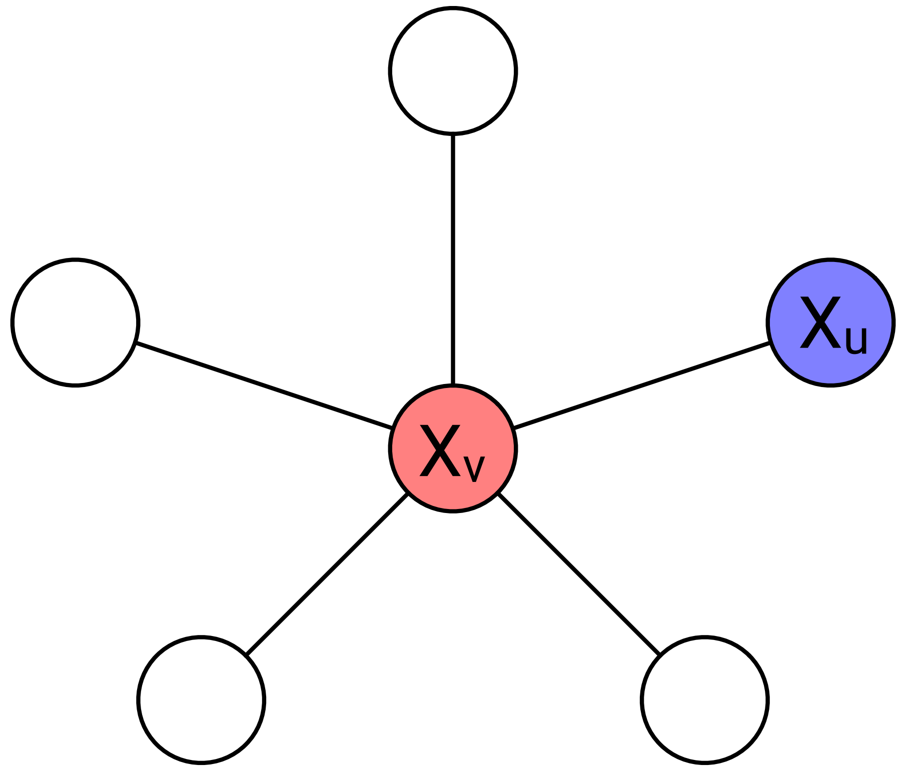

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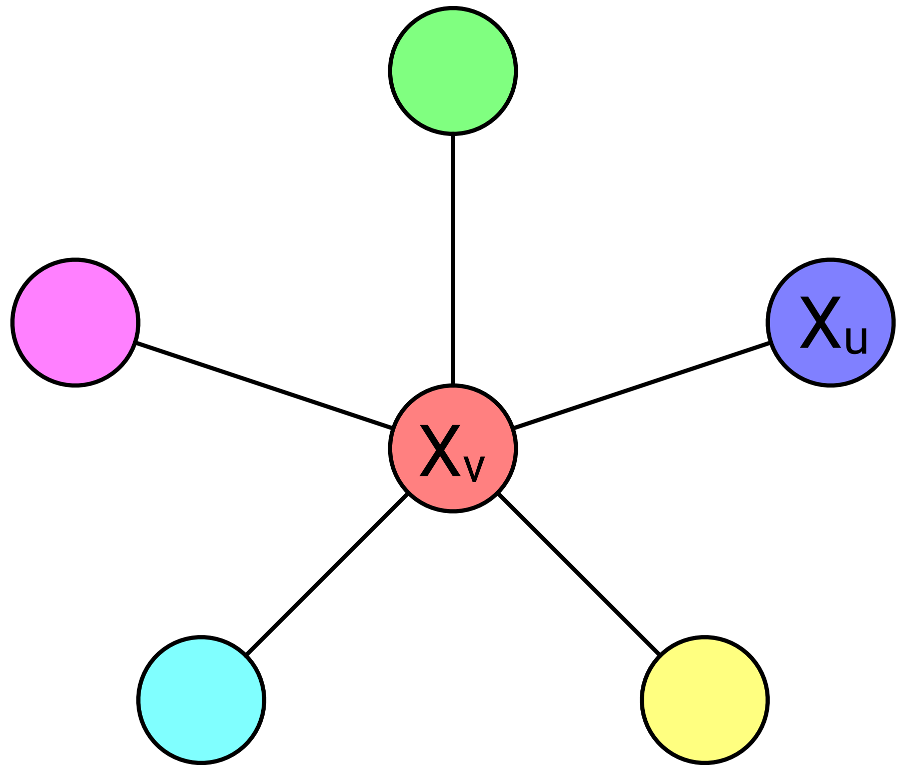

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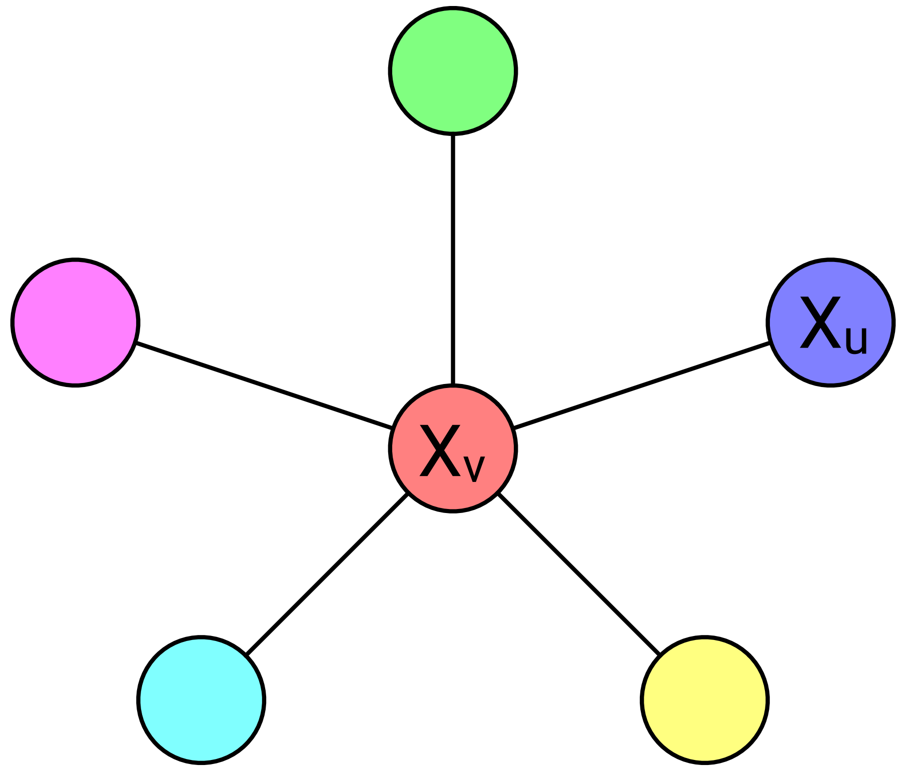

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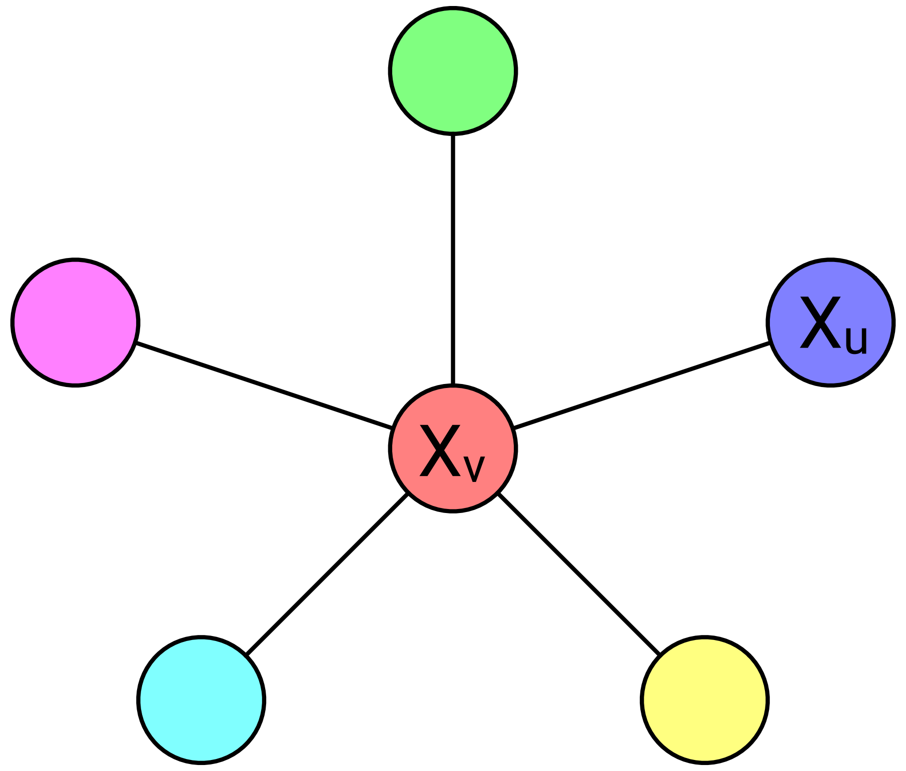

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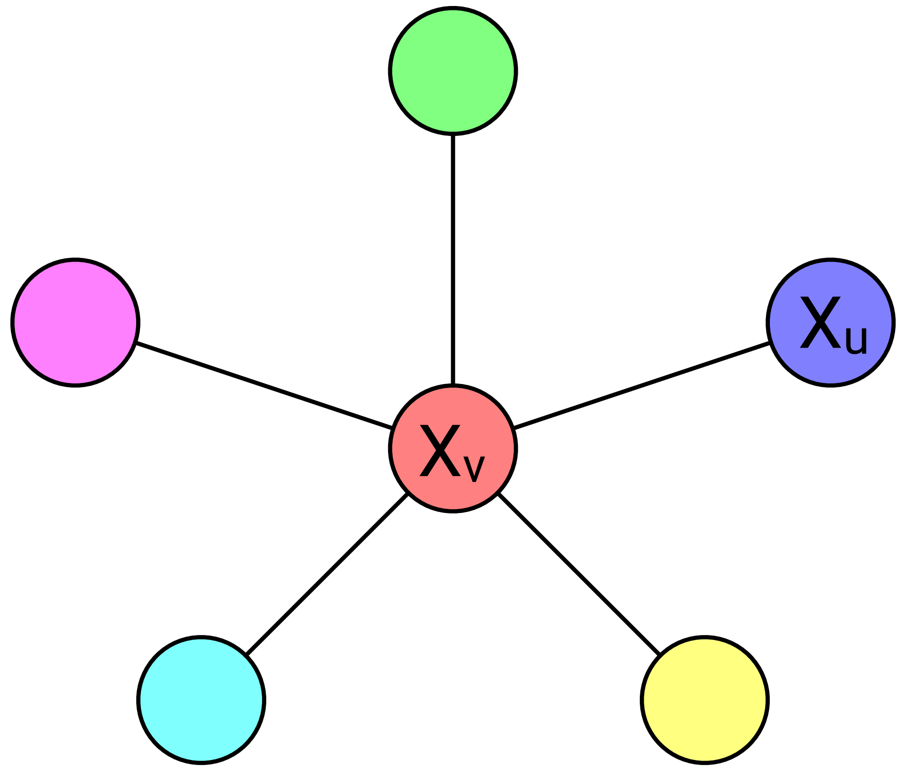

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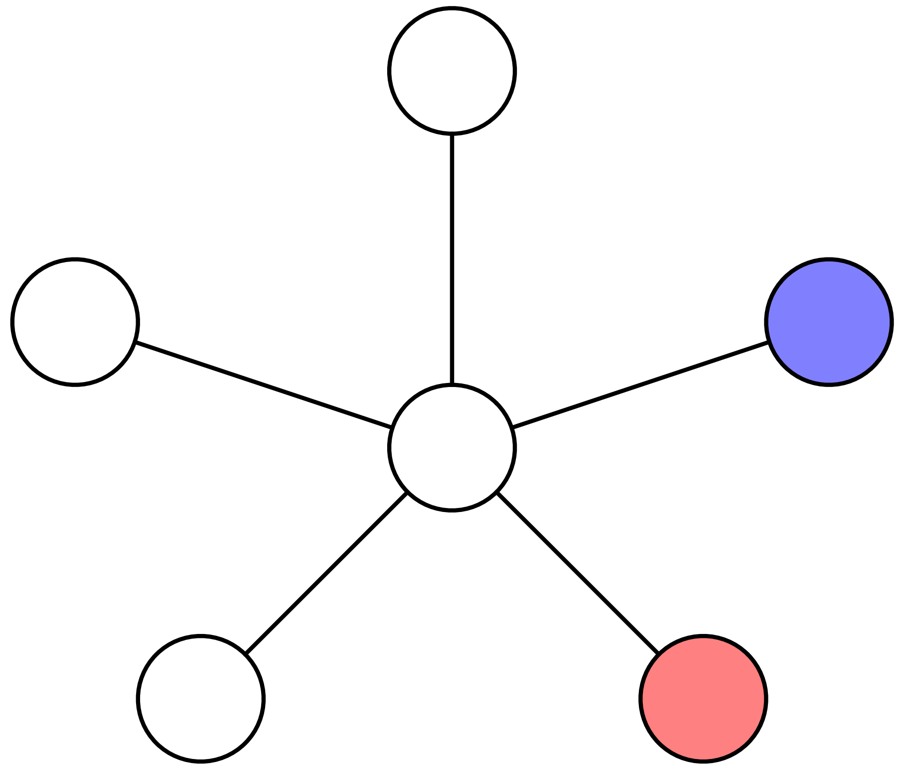

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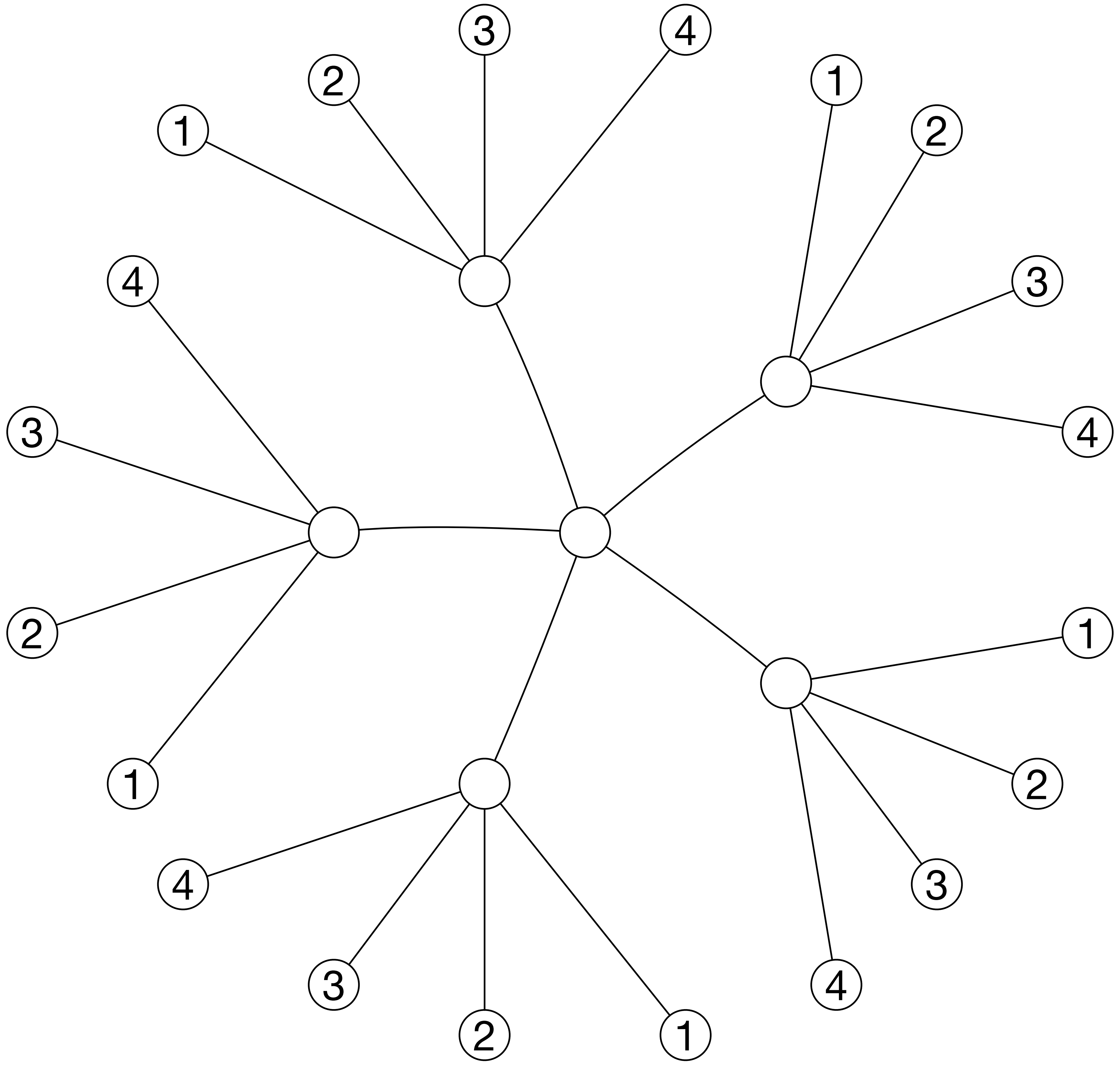

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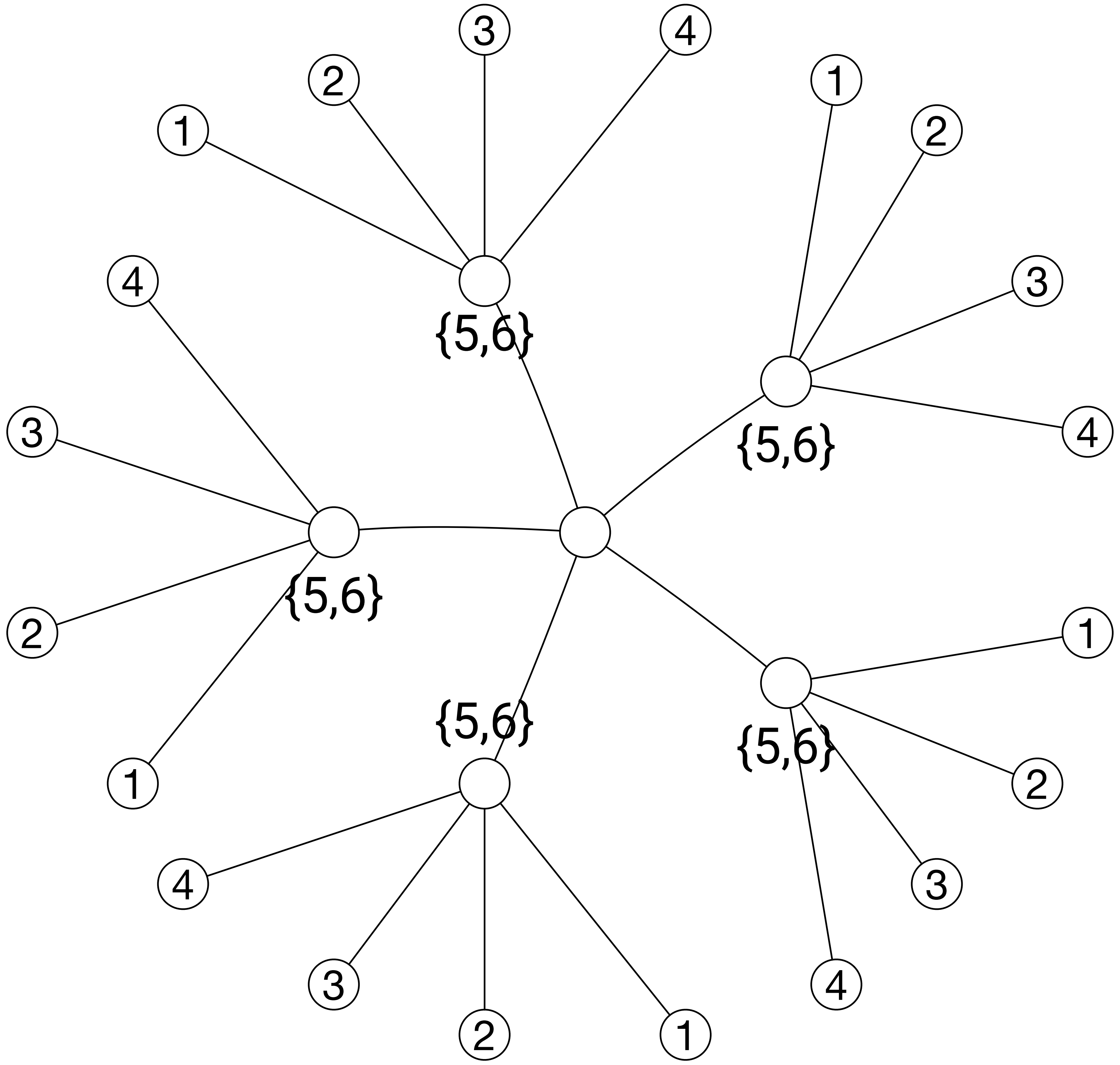

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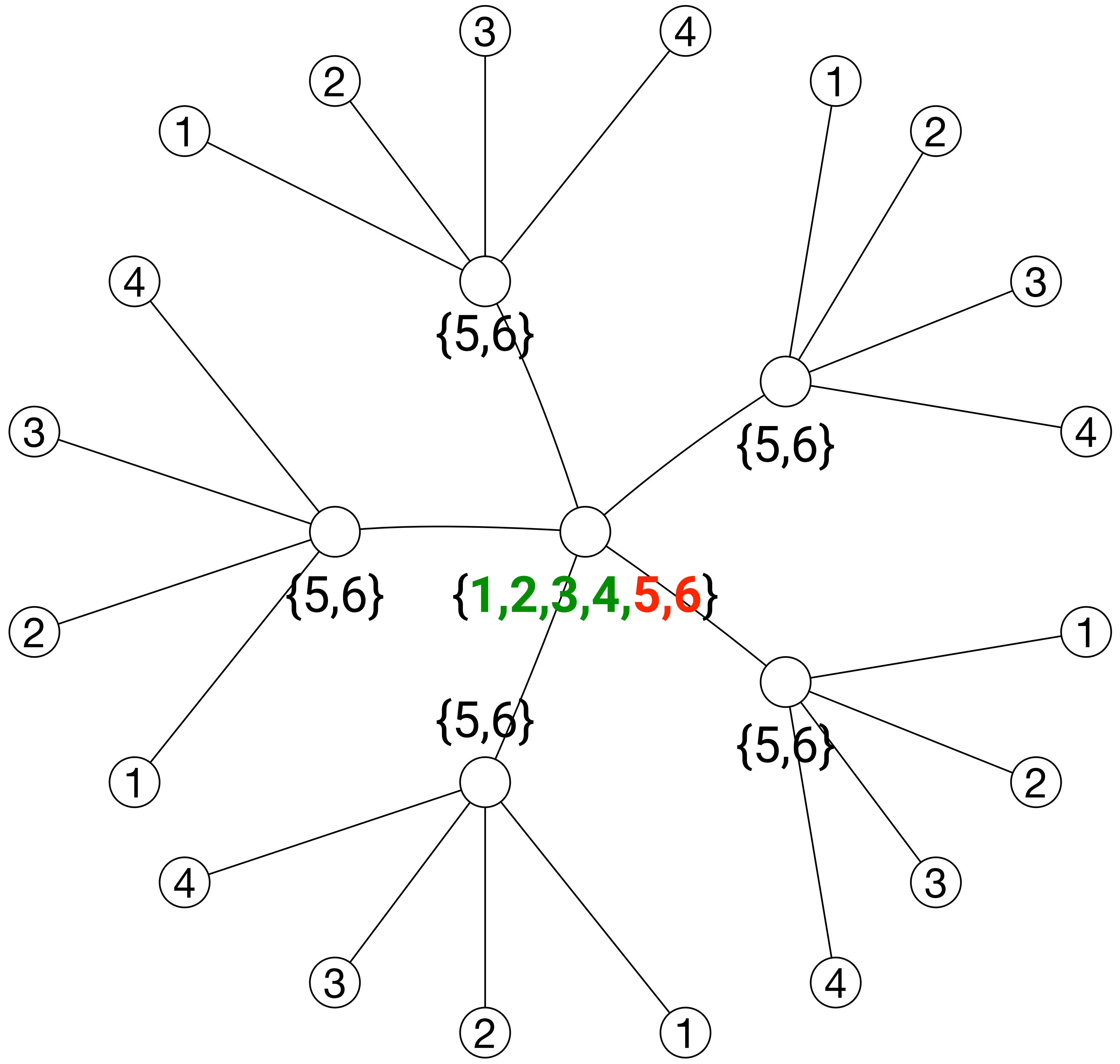
- **Is it still true** that for every possible color X_v, the probability that $X_v \neq X_u$ for all neighbors u of v is at least 1/e?
  - **No!** Some colors are better than others.



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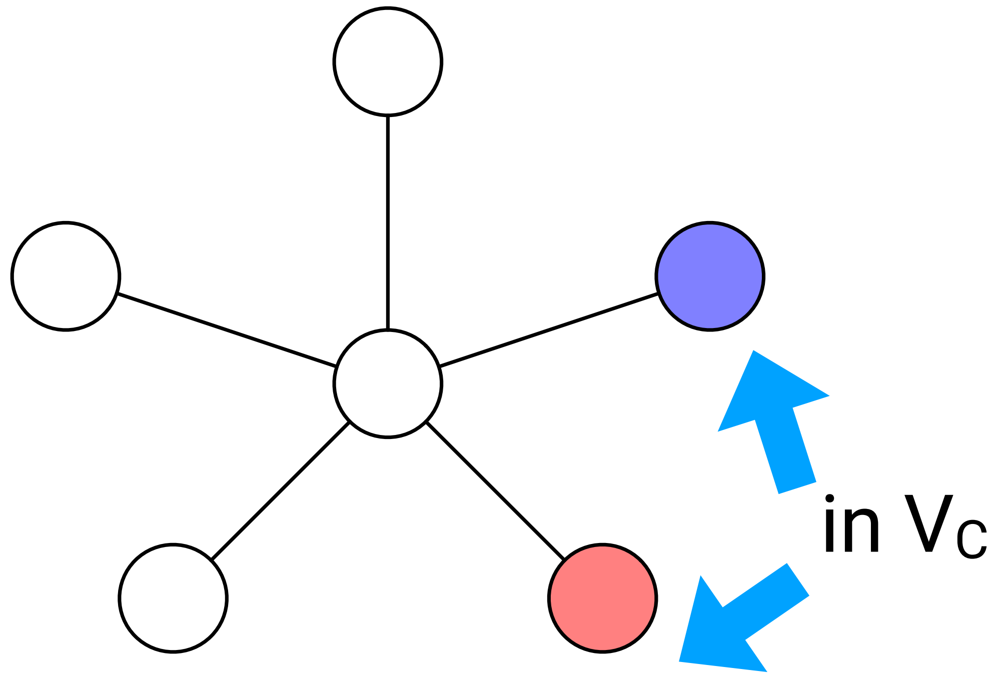

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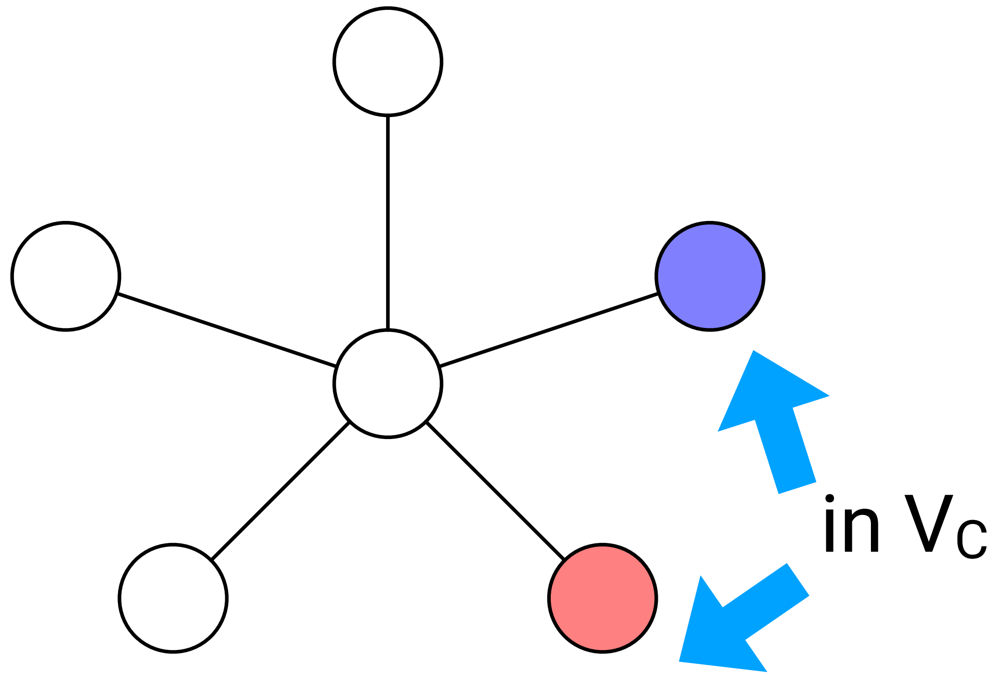
Subgraph induced by nodes in V_C



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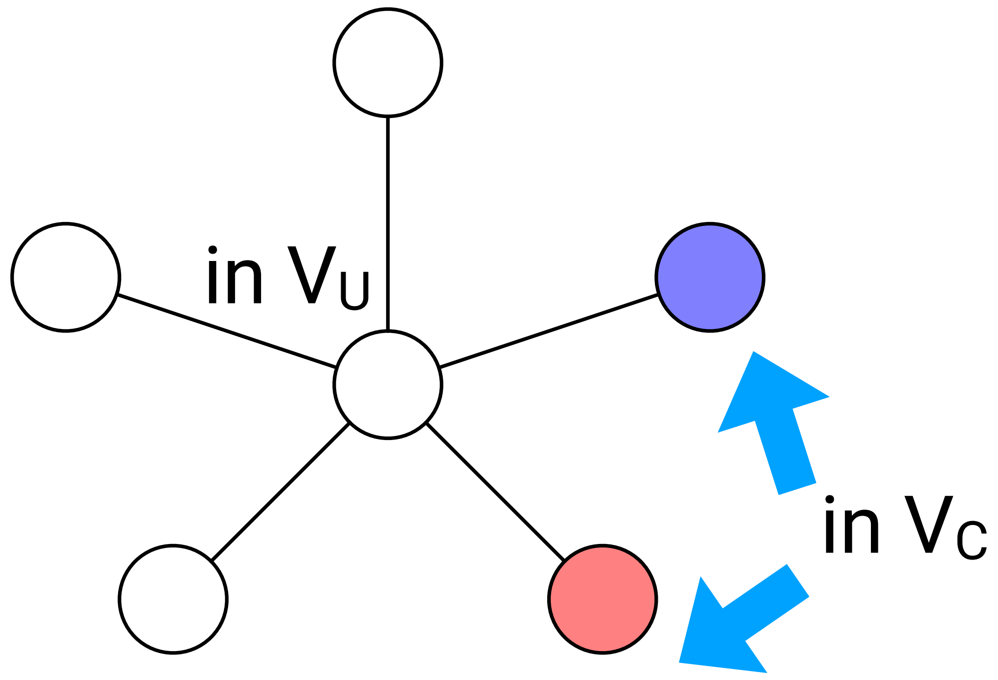

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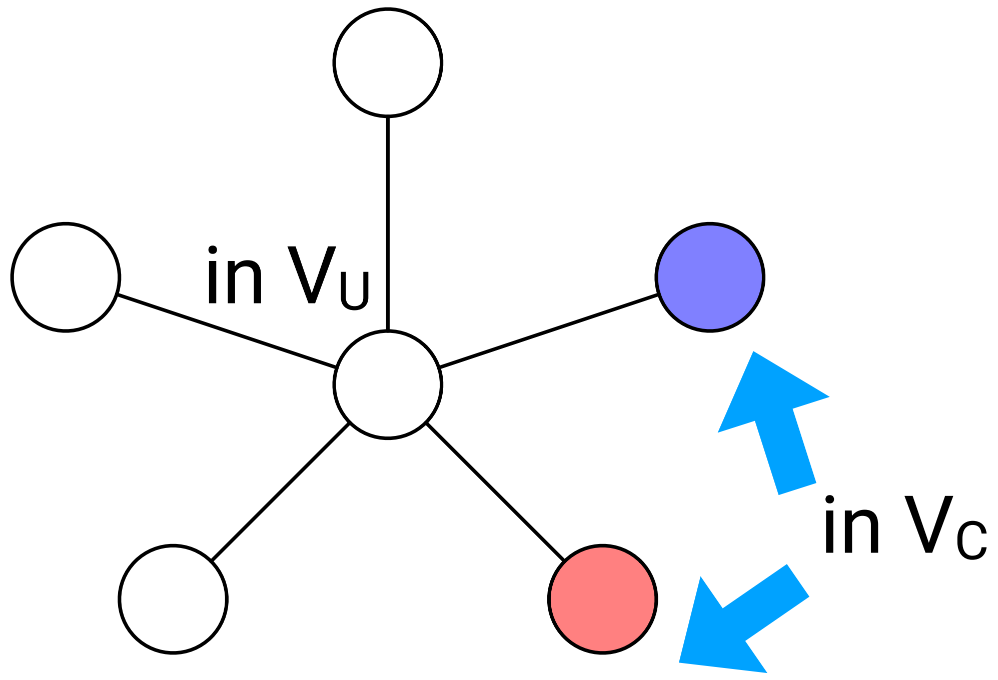

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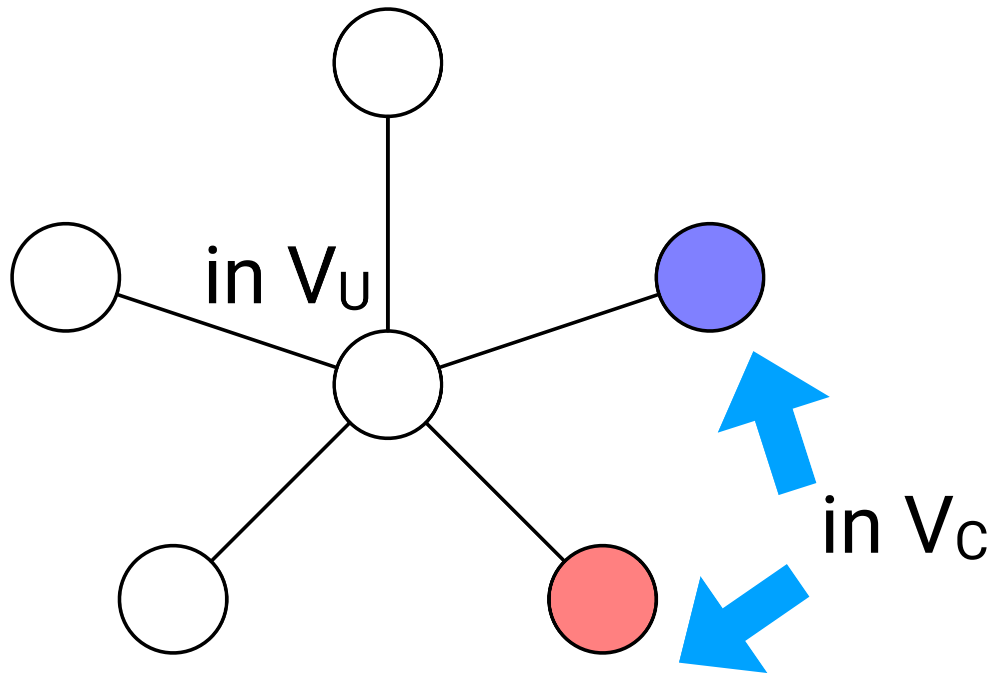

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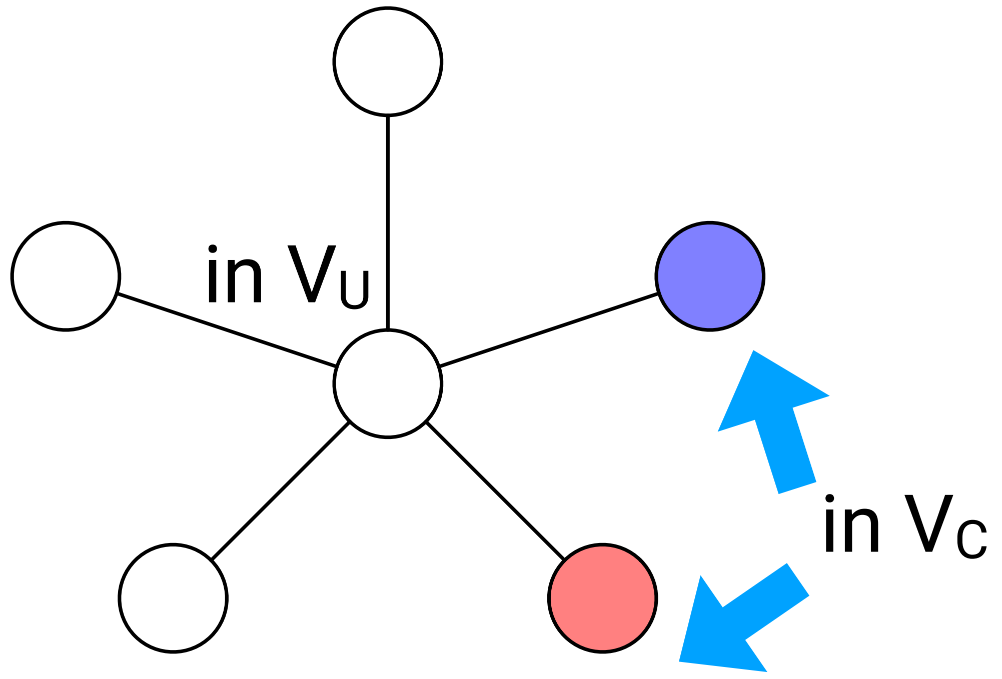

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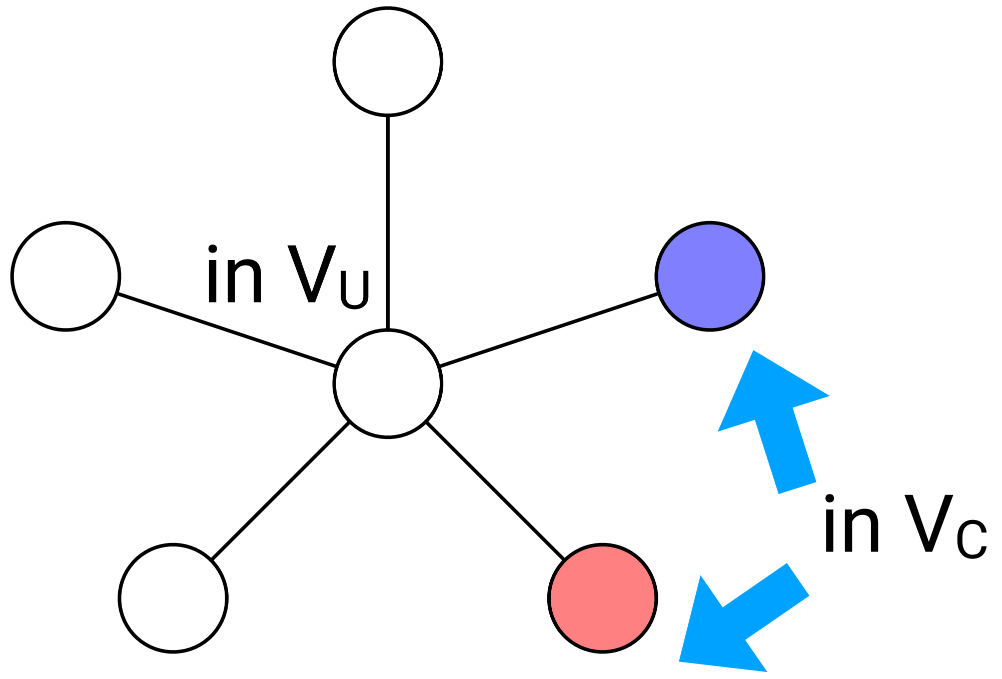

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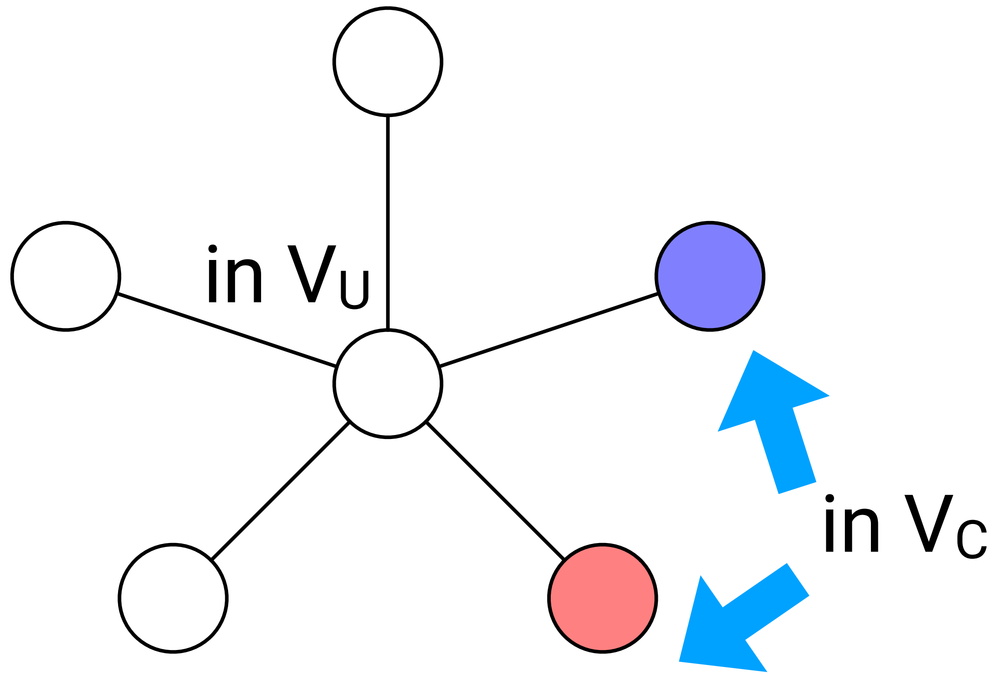

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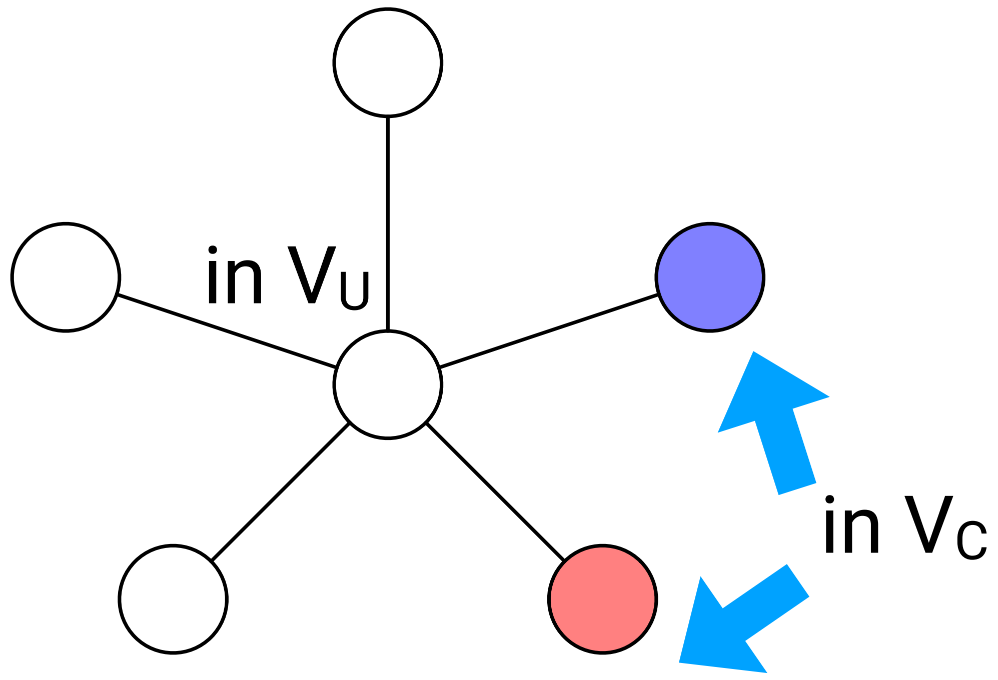

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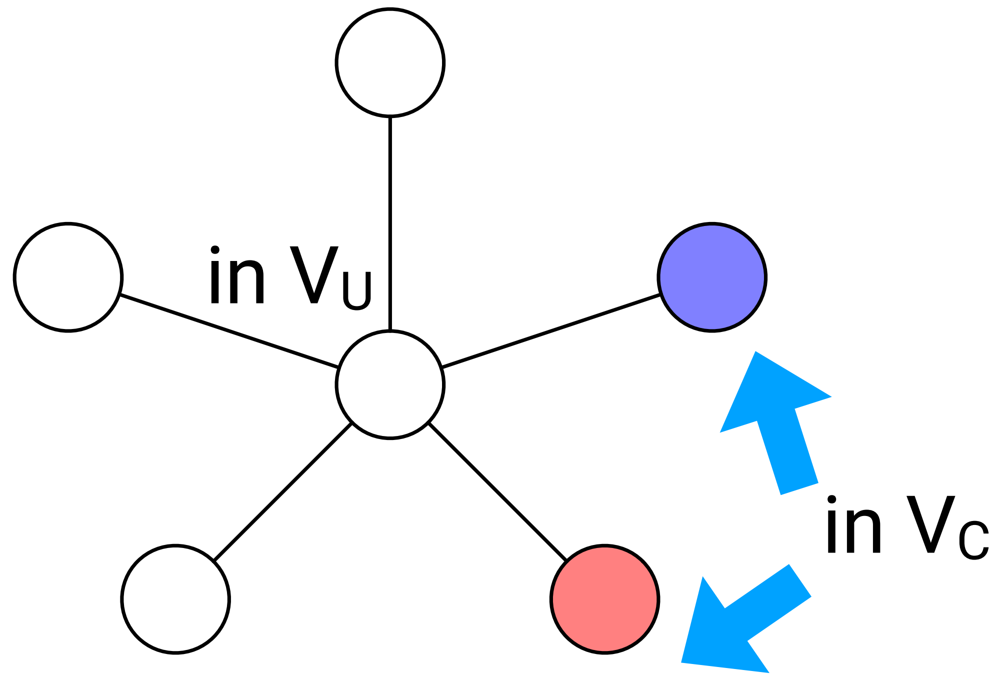

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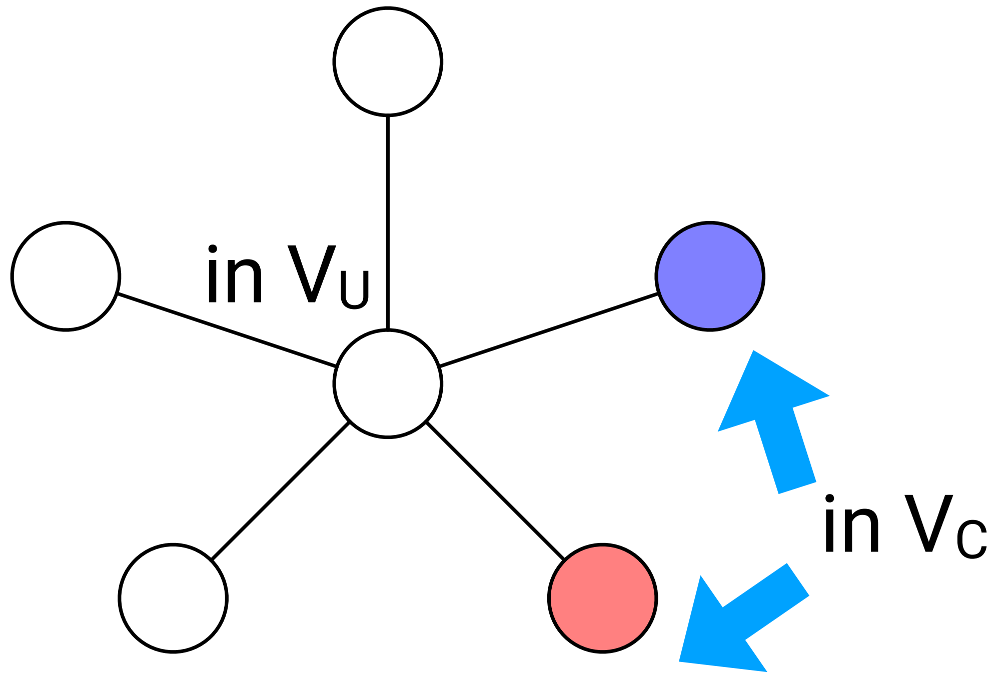

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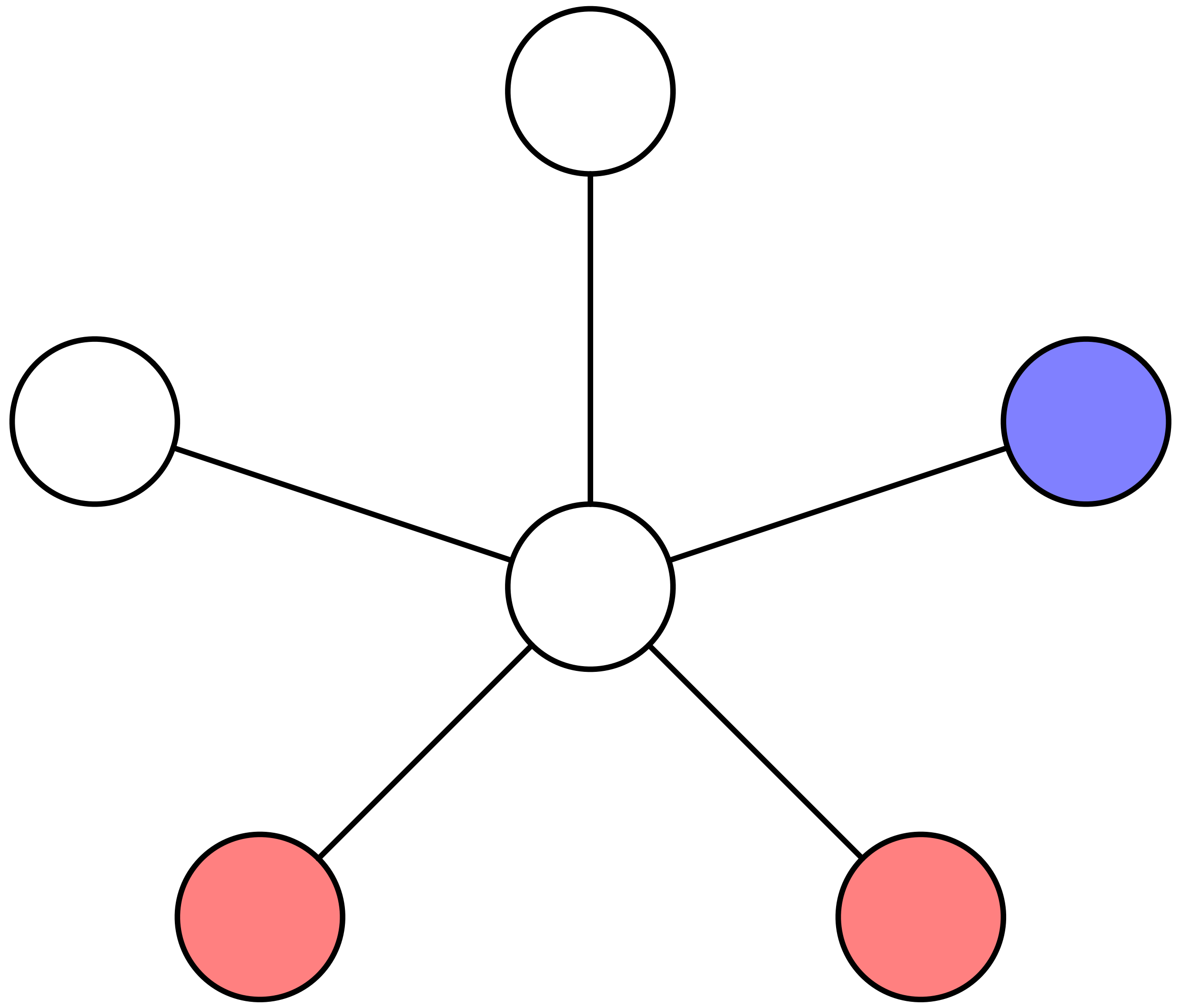

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$|F_u| \geq 2$, otherwise u has no uncolored neighbors

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Probability that u does **not** pick x

Neighbors that could also pick x

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$|F_u| \geq 2$, otherwise u has no uncolored neighbors

Extending an Existing Coloring

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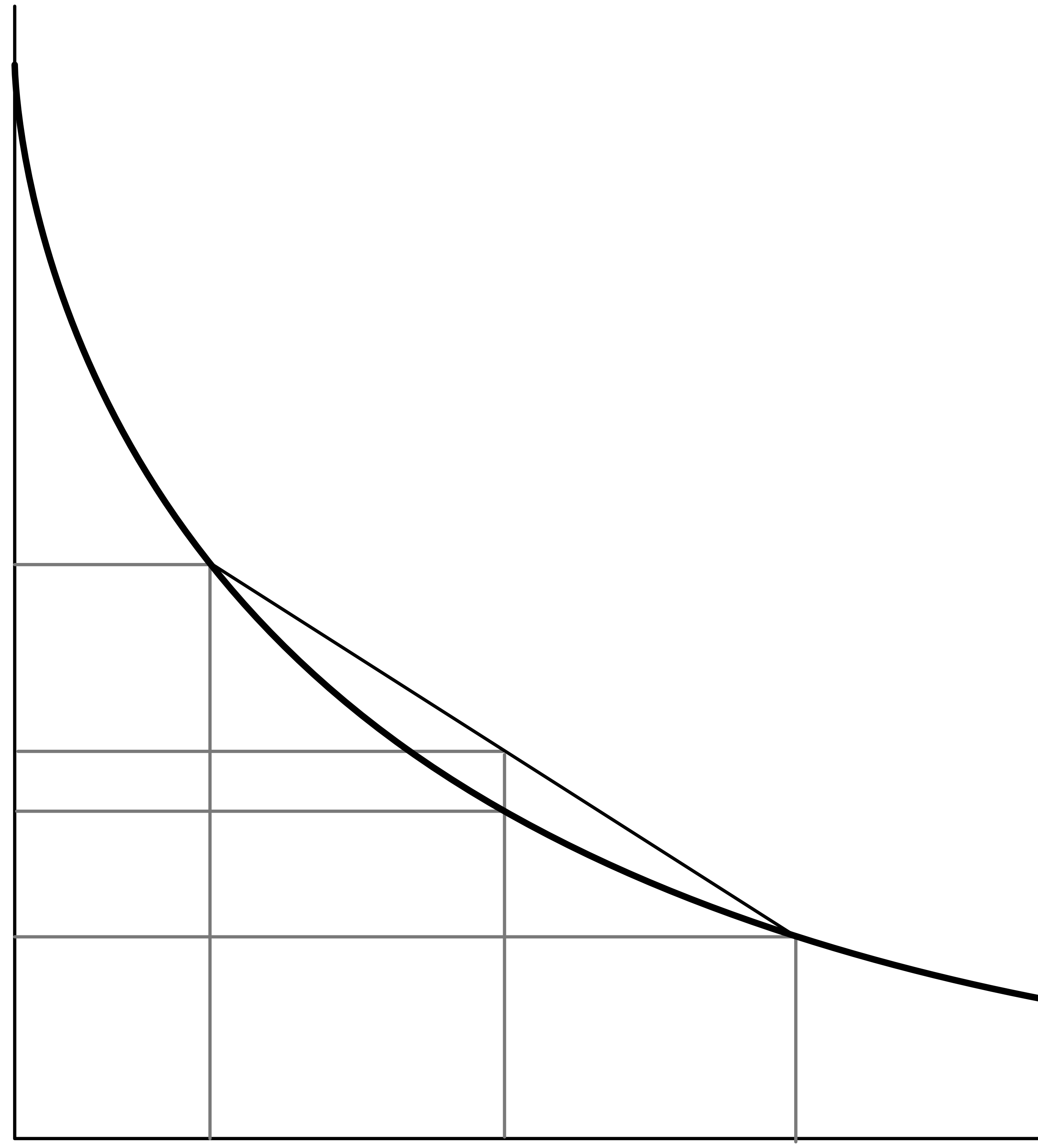

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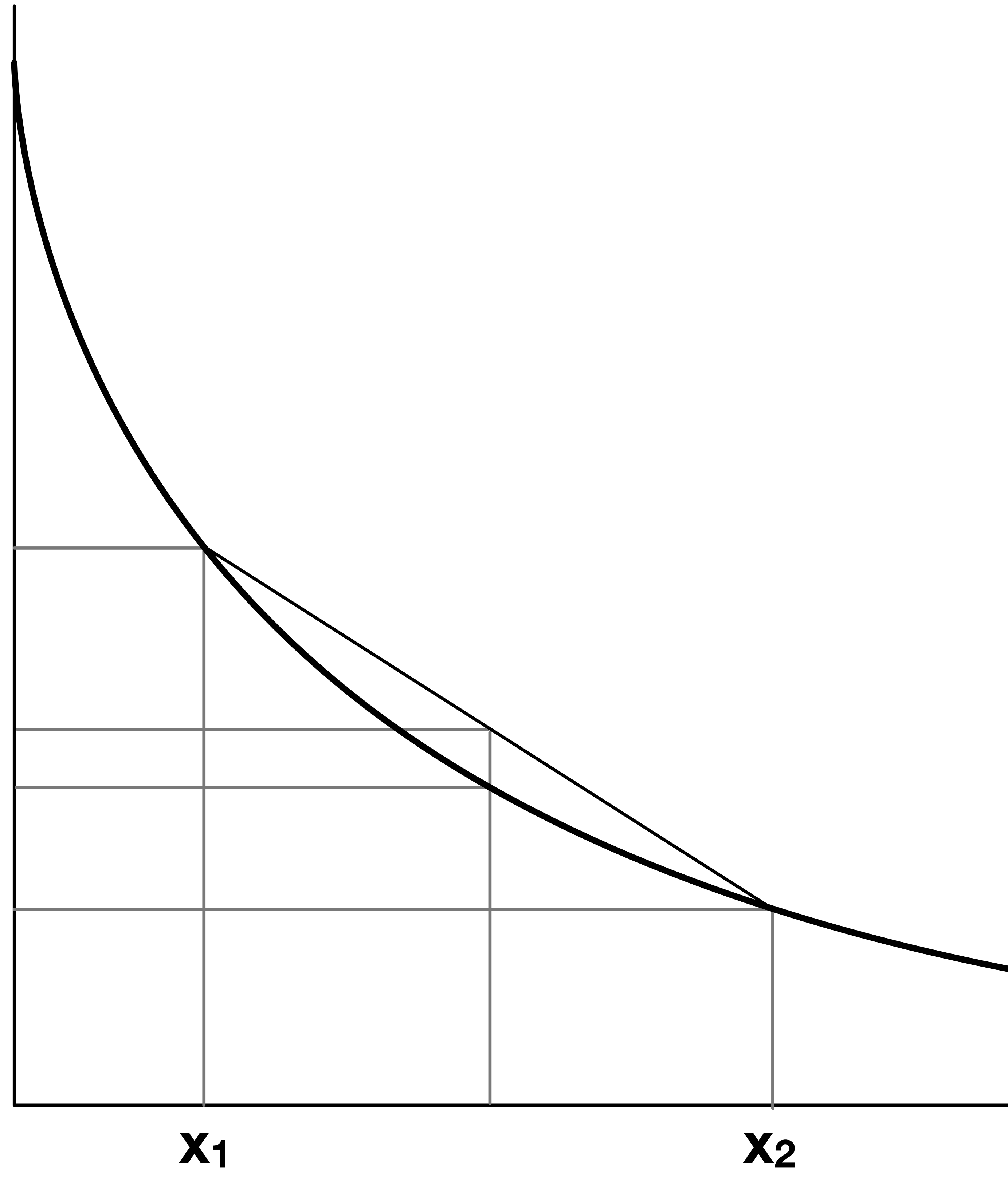

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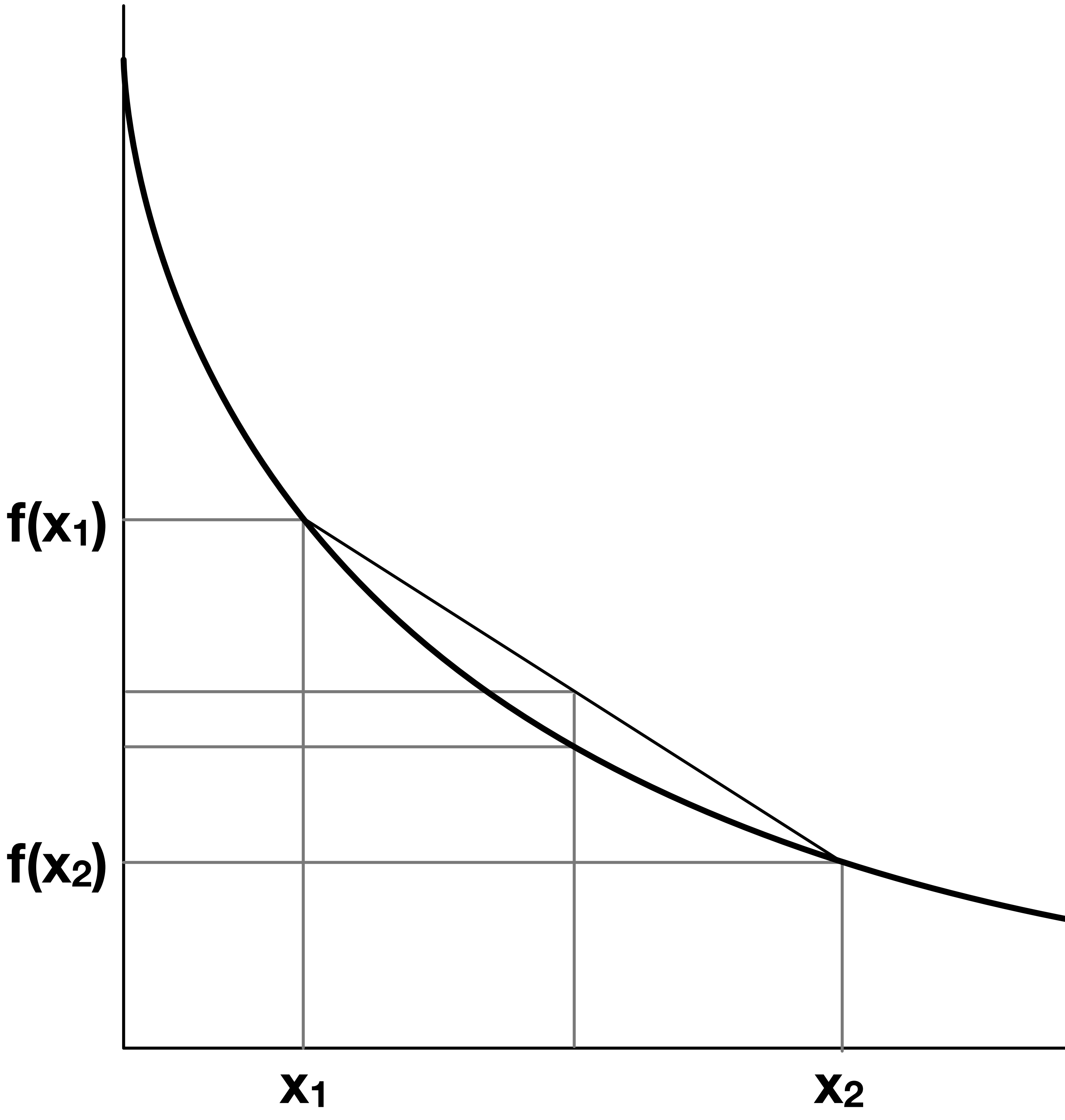

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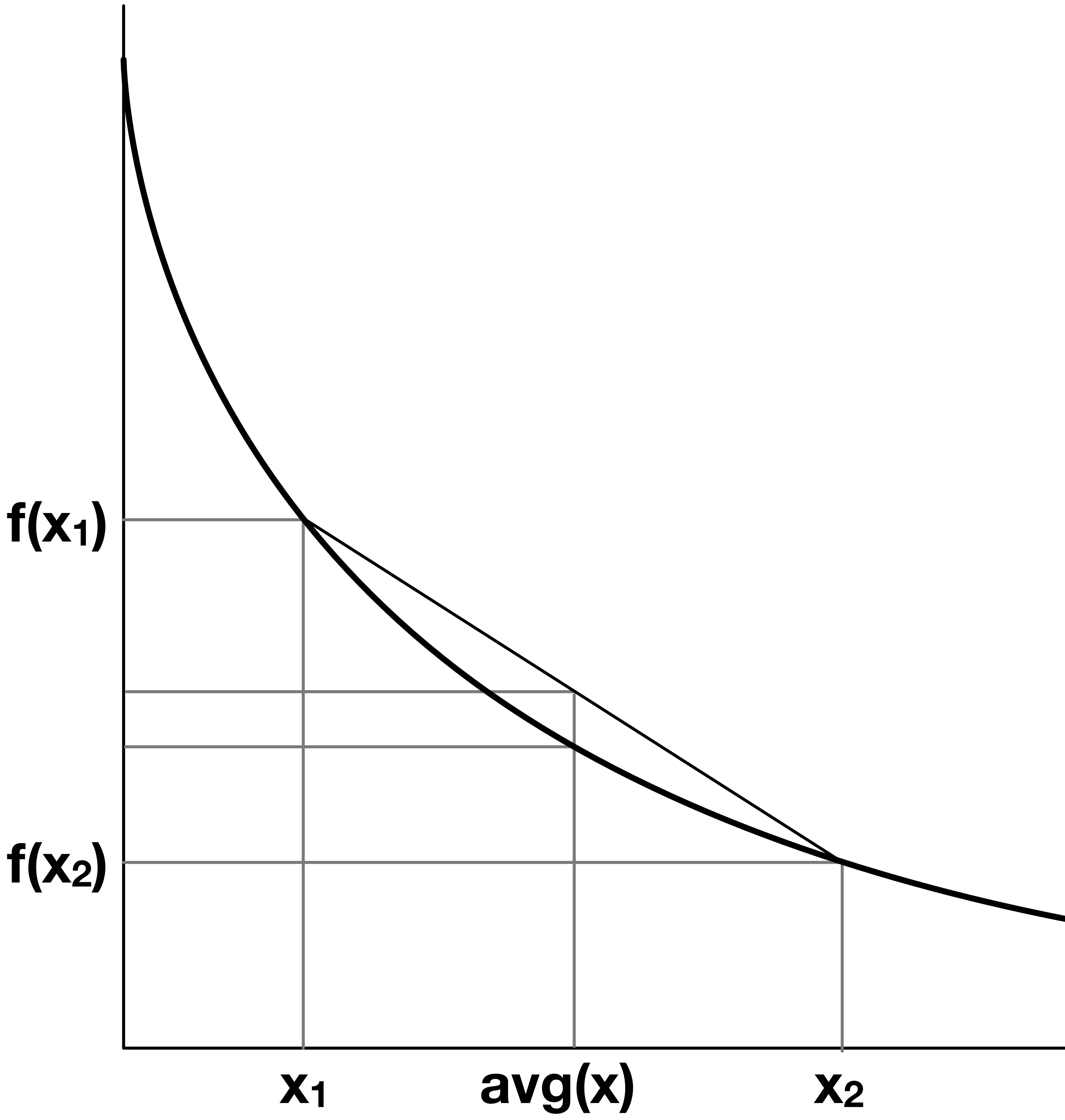

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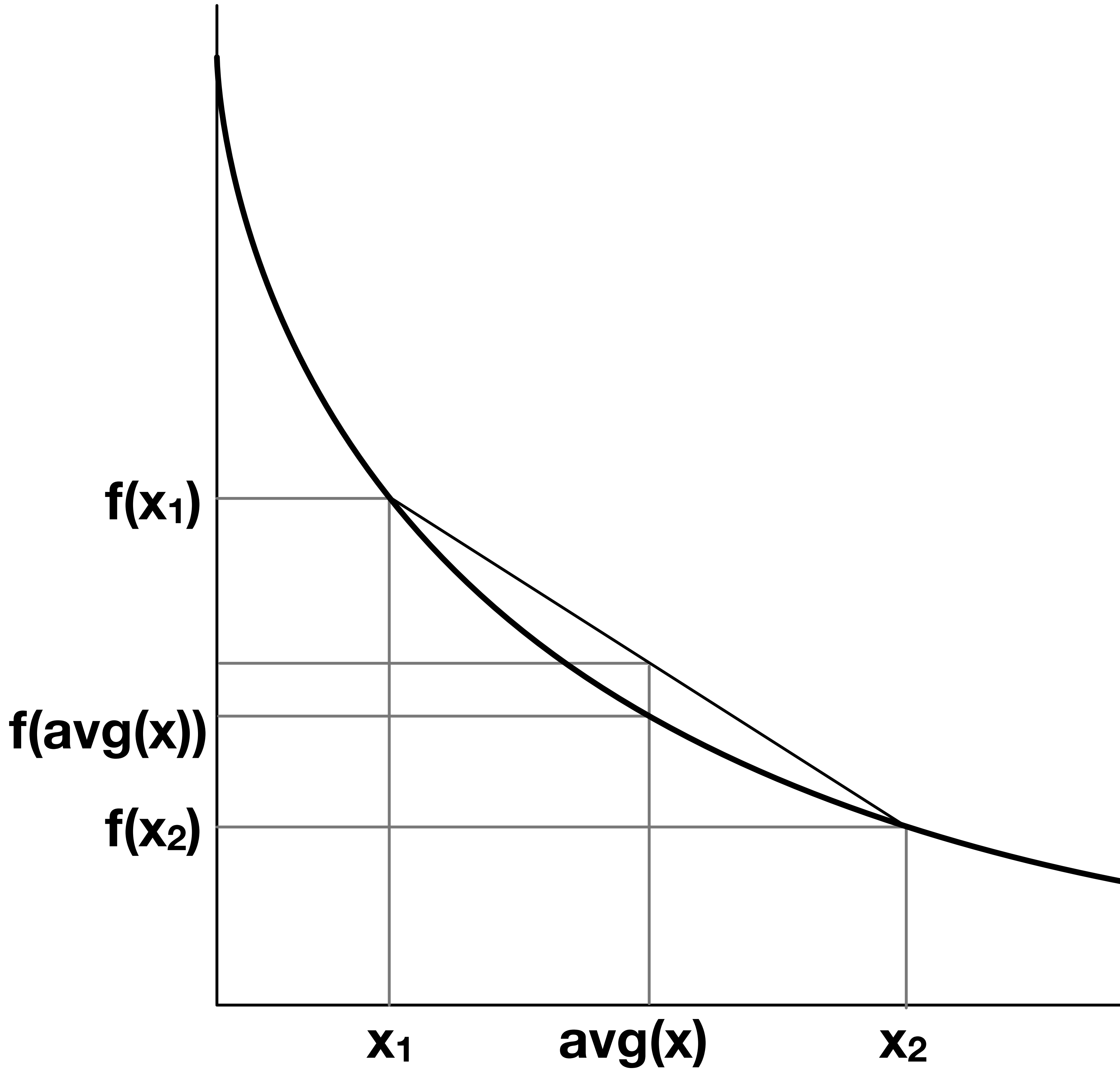

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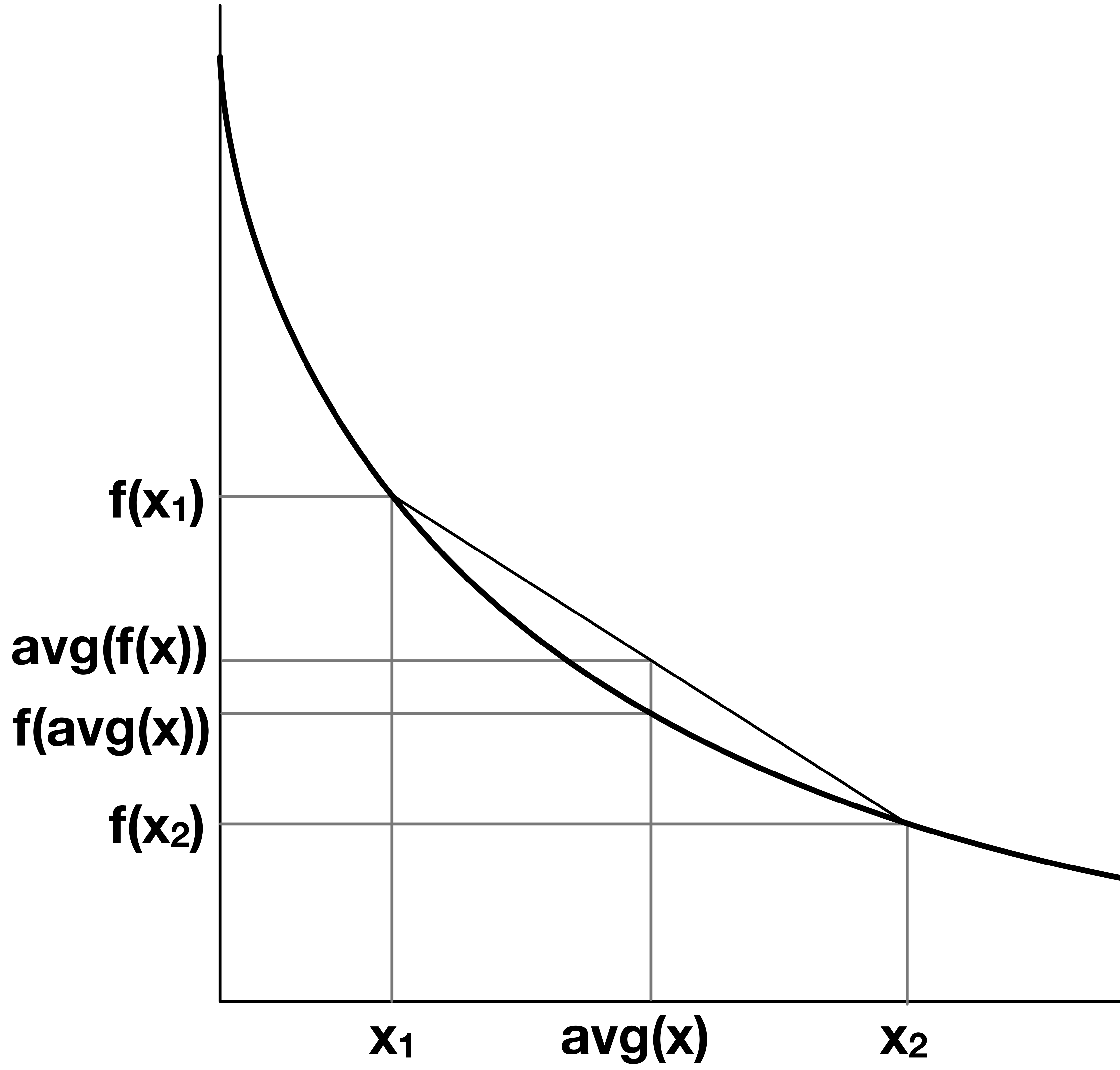

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Union Bound: $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

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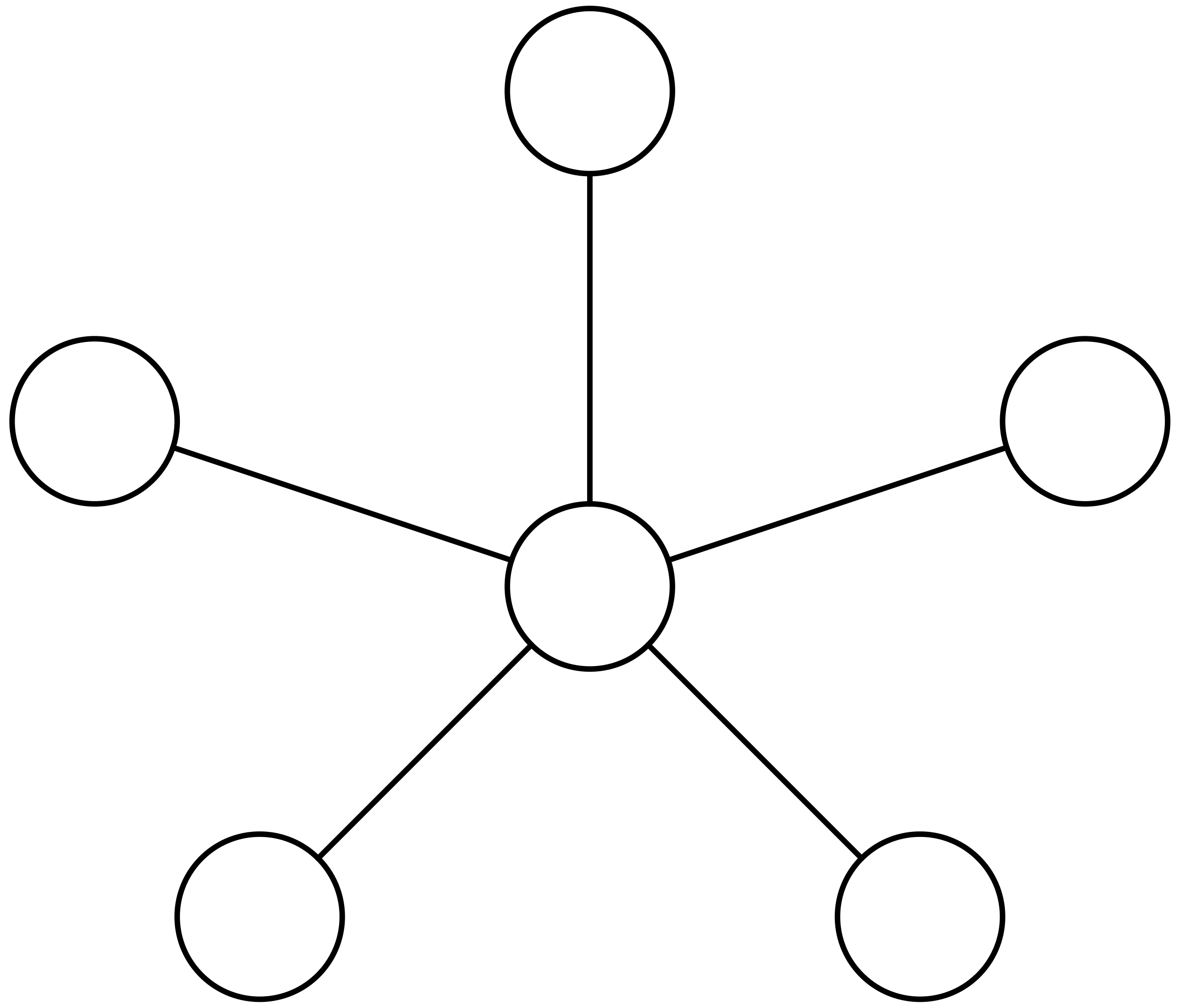

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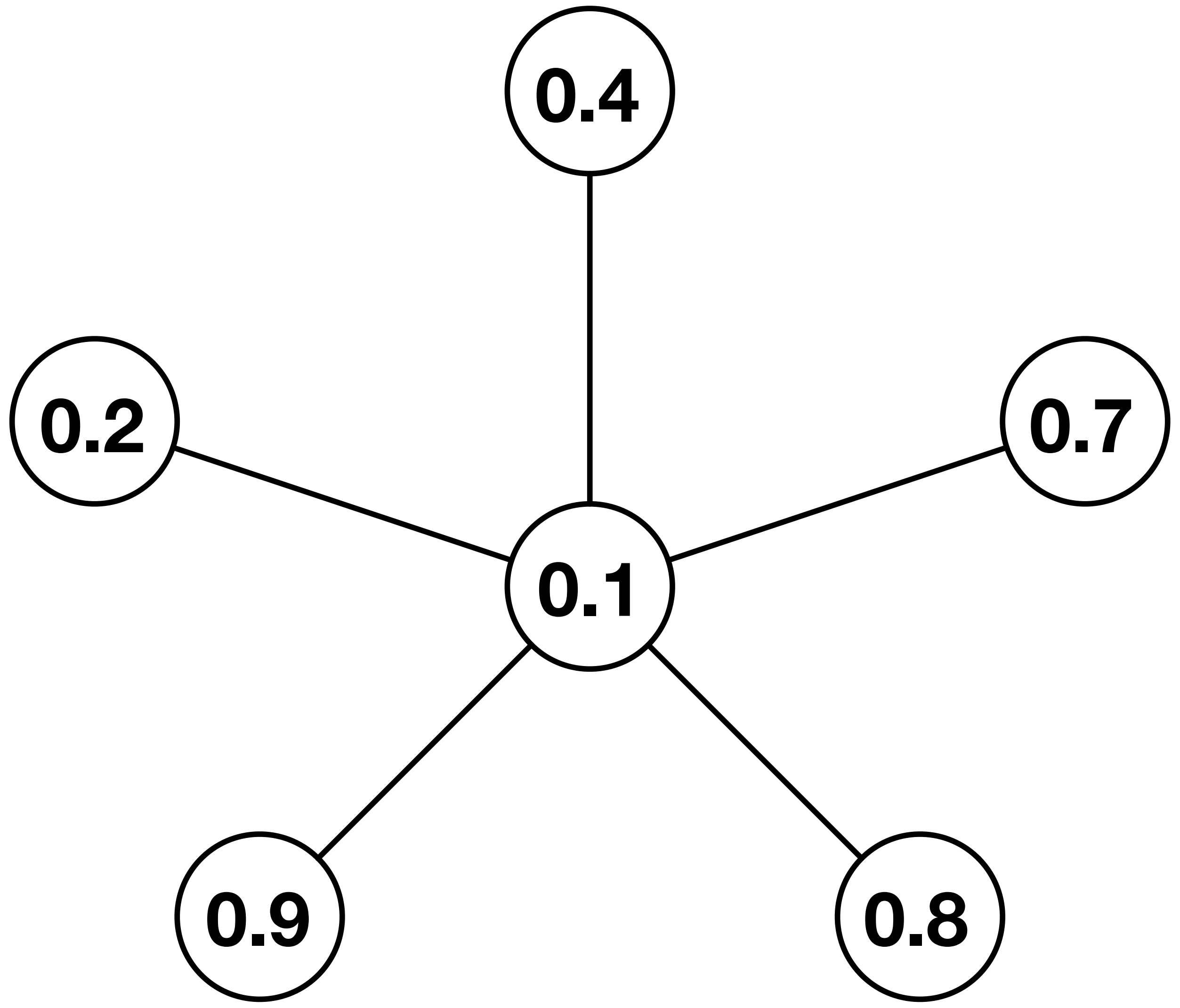

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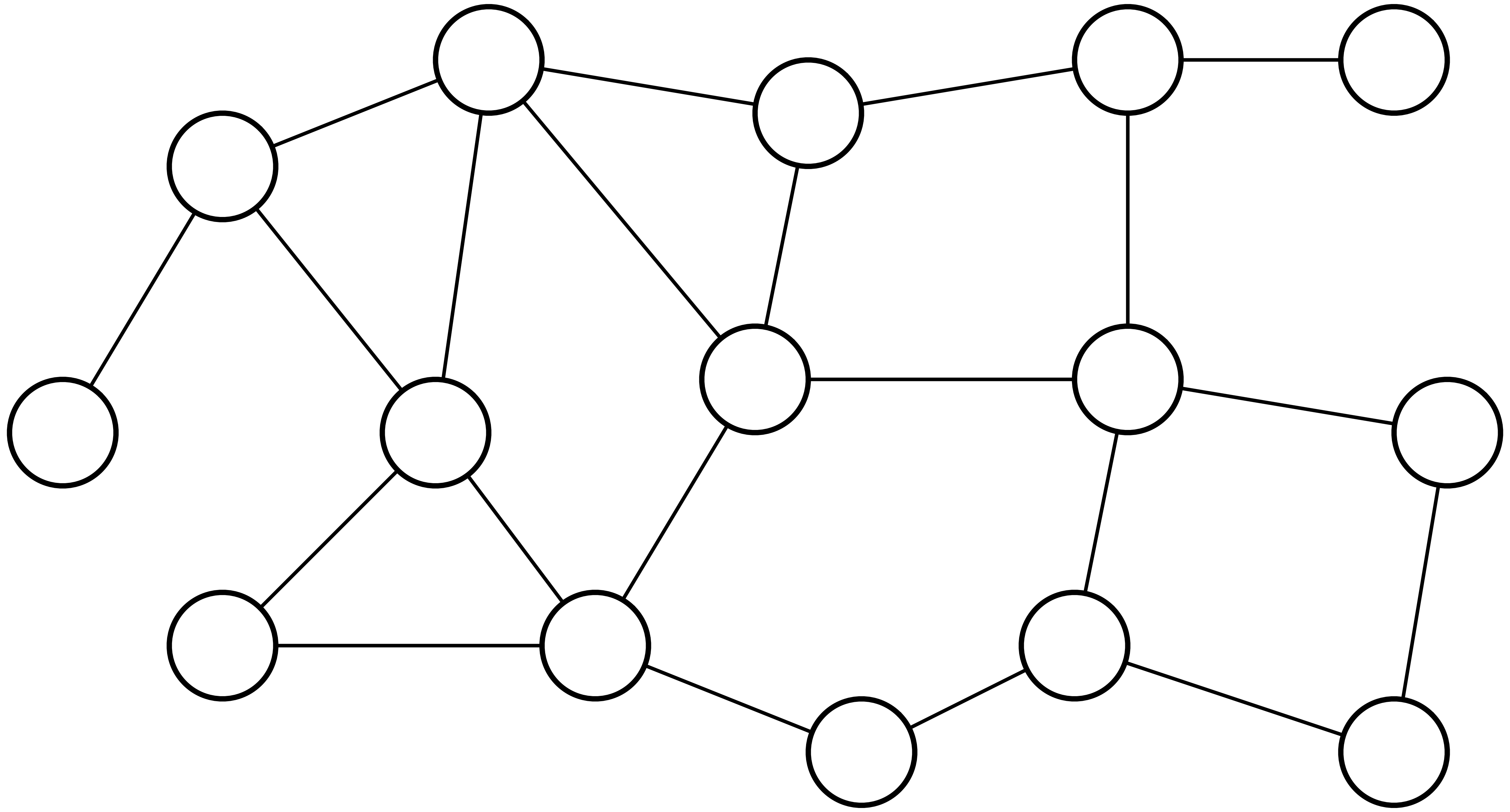

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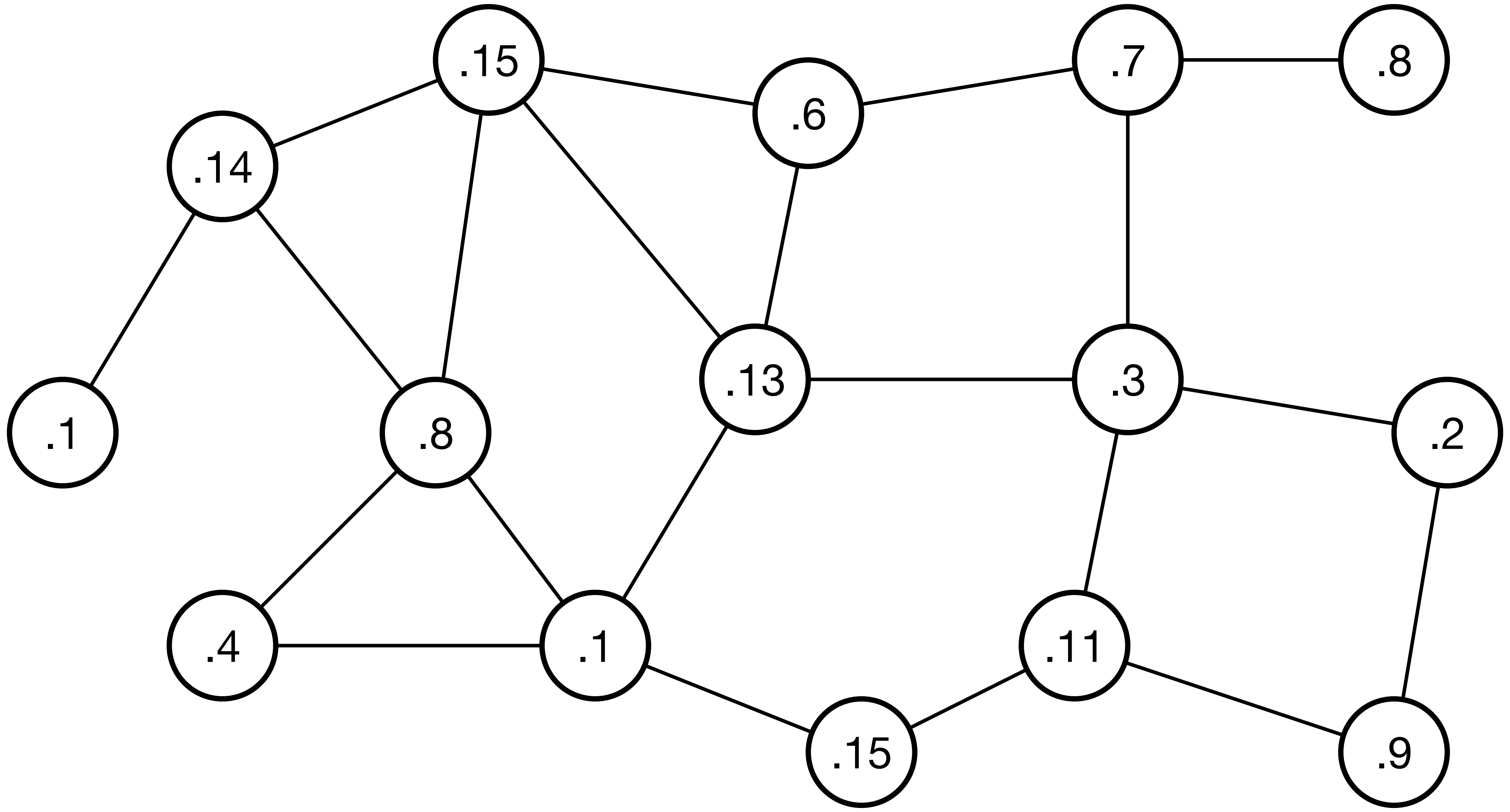

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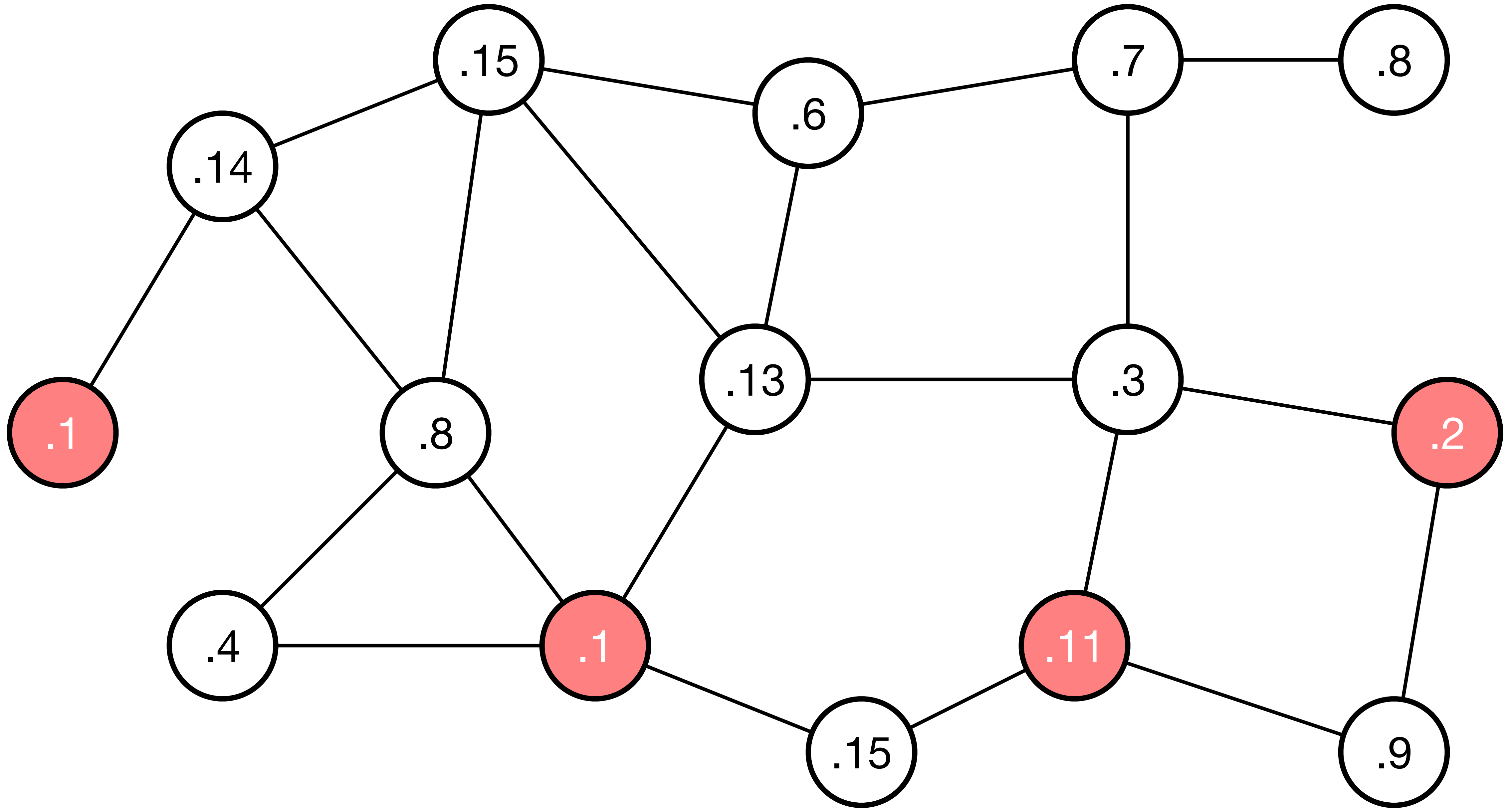

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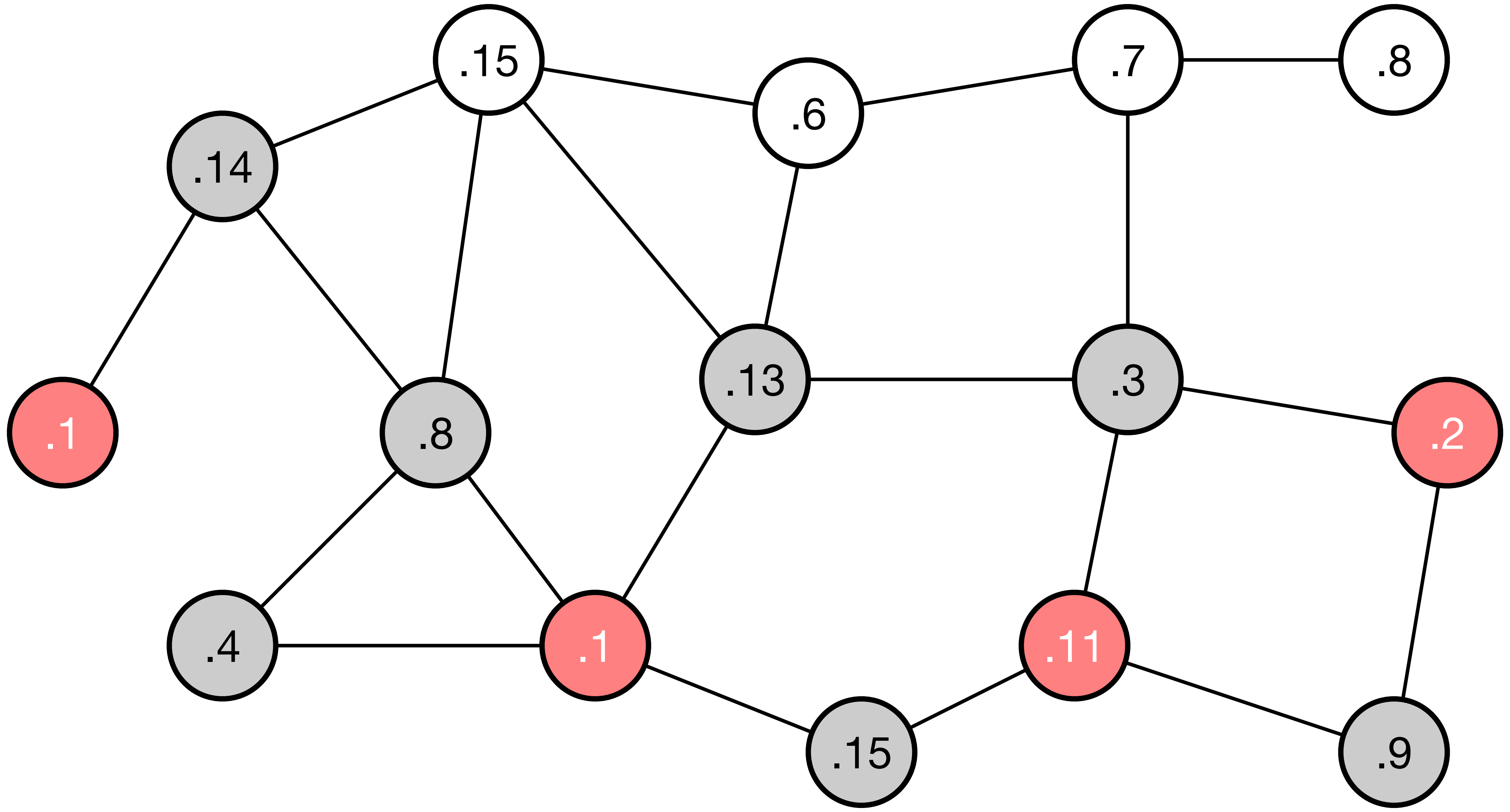

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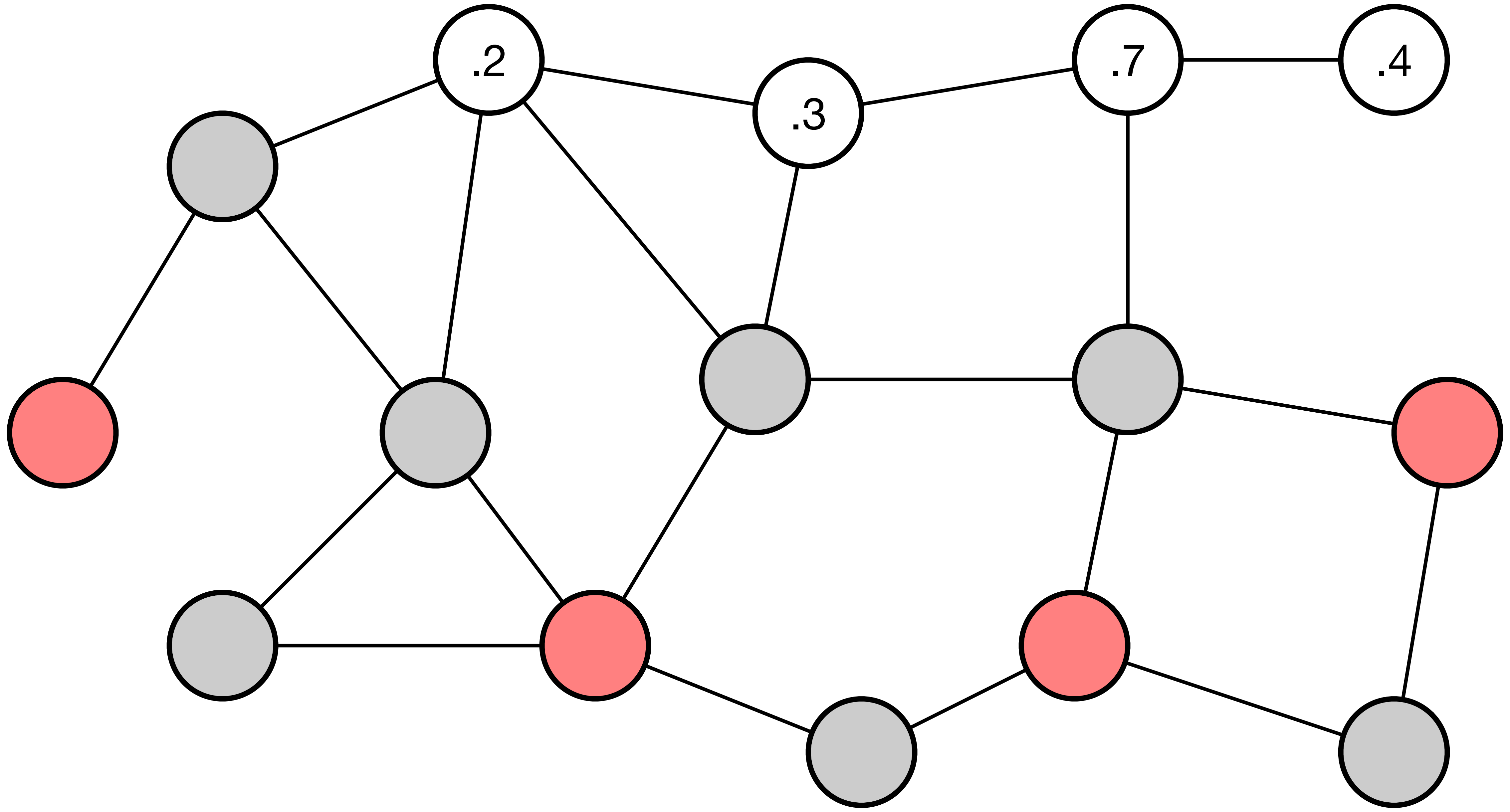

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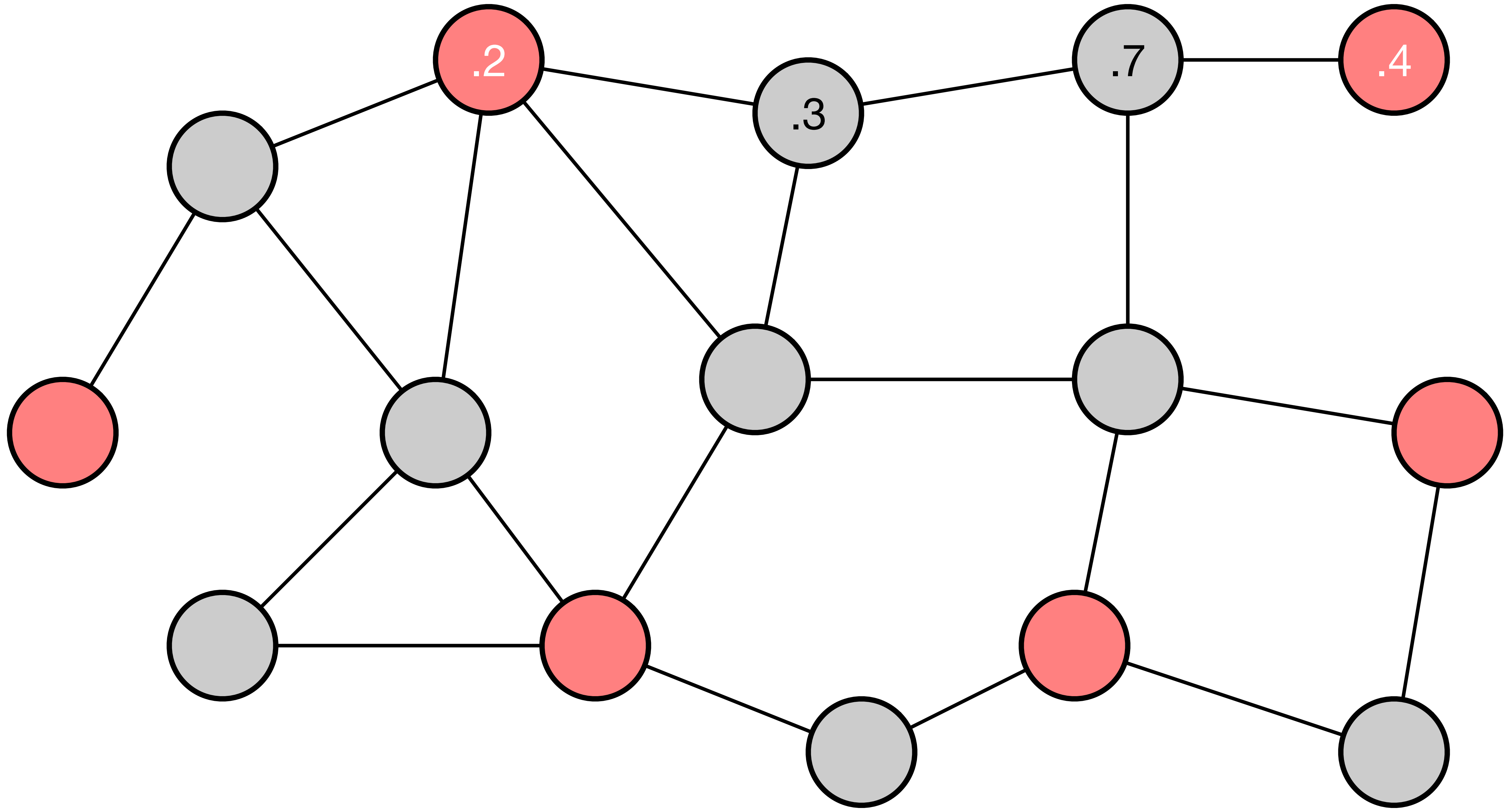

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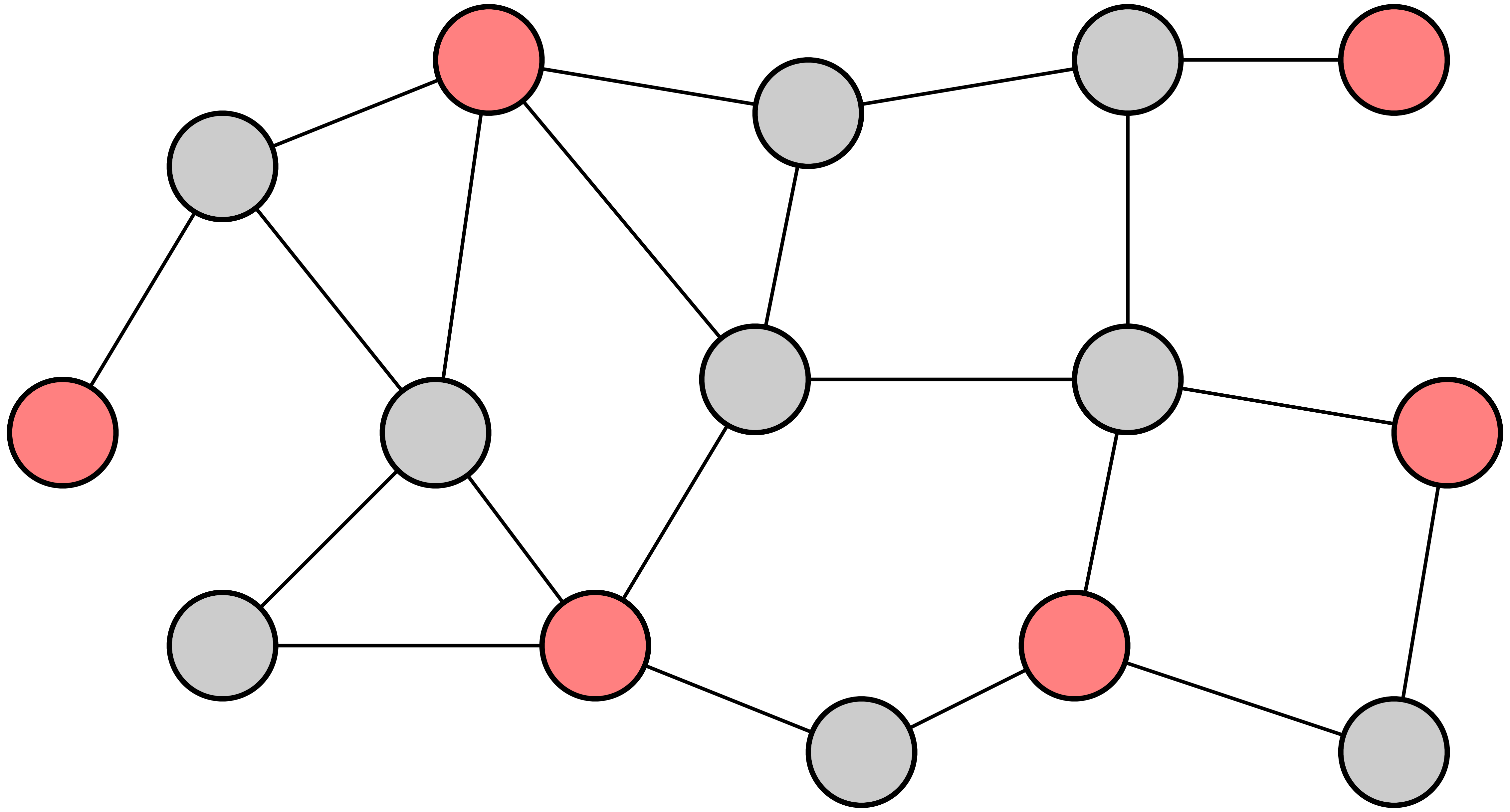

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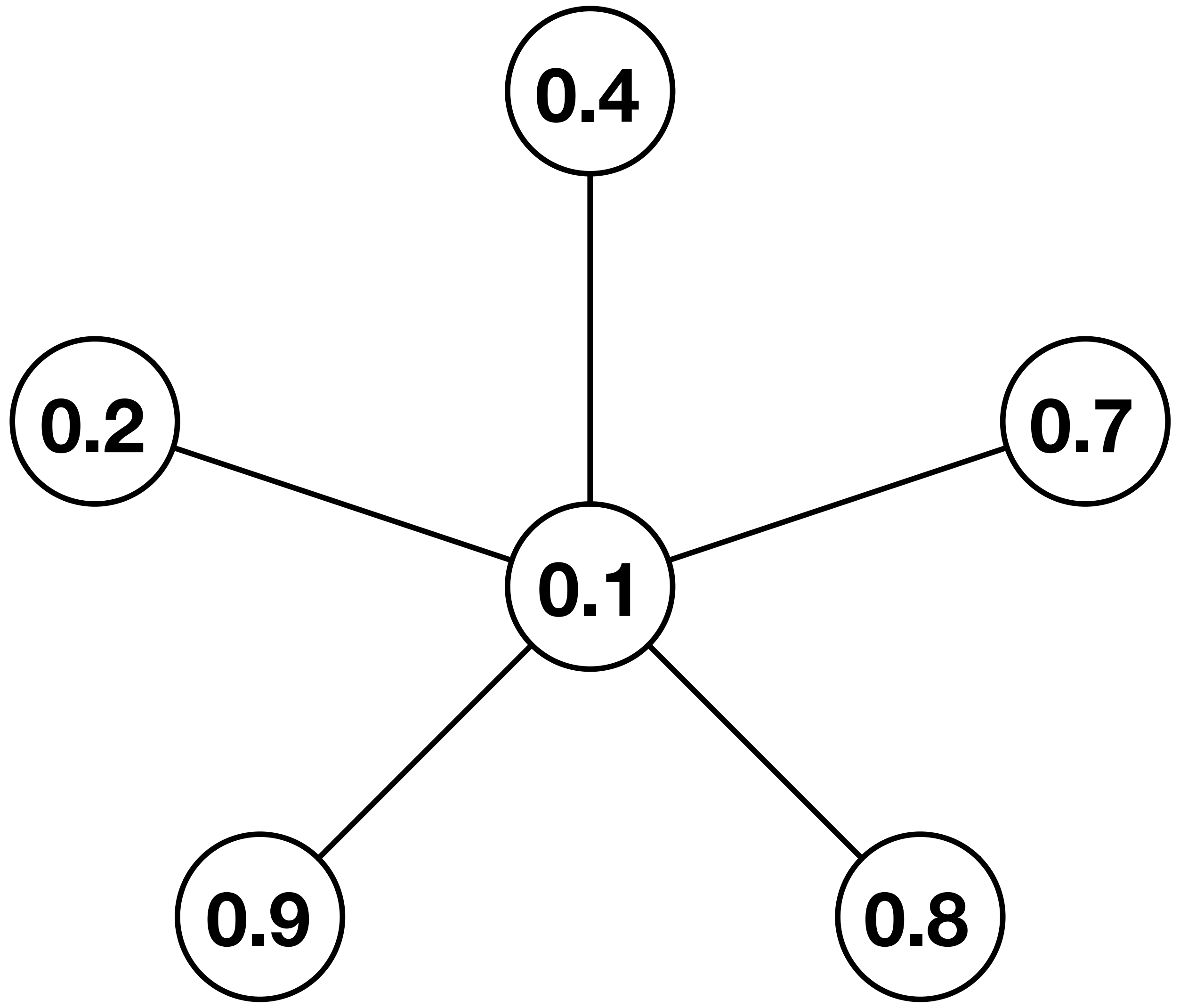

Randomized MIS: Luby's Algorithm

- At each round:
  - Each node v picks a random number **$R_v \in [0,1]$**
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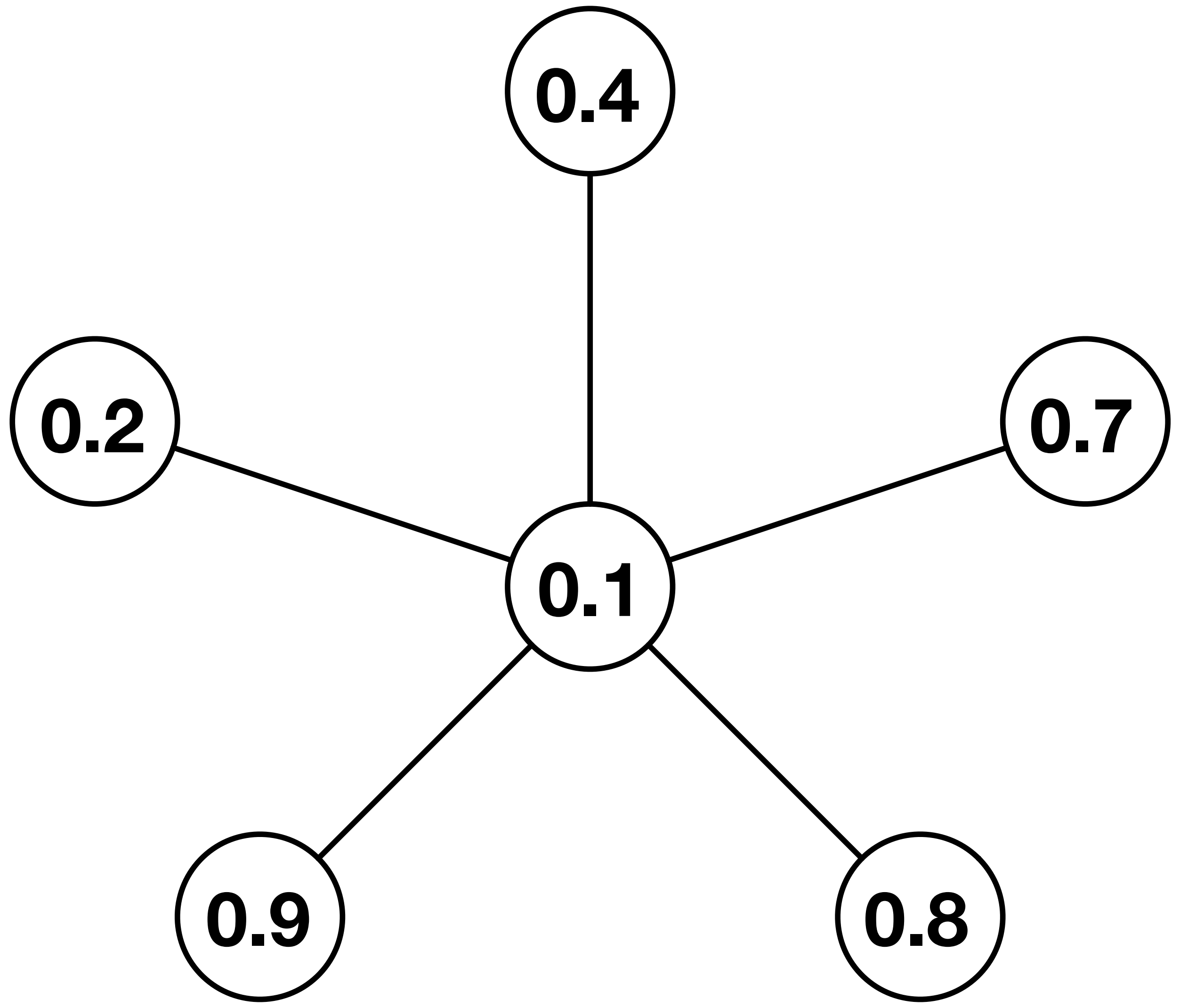

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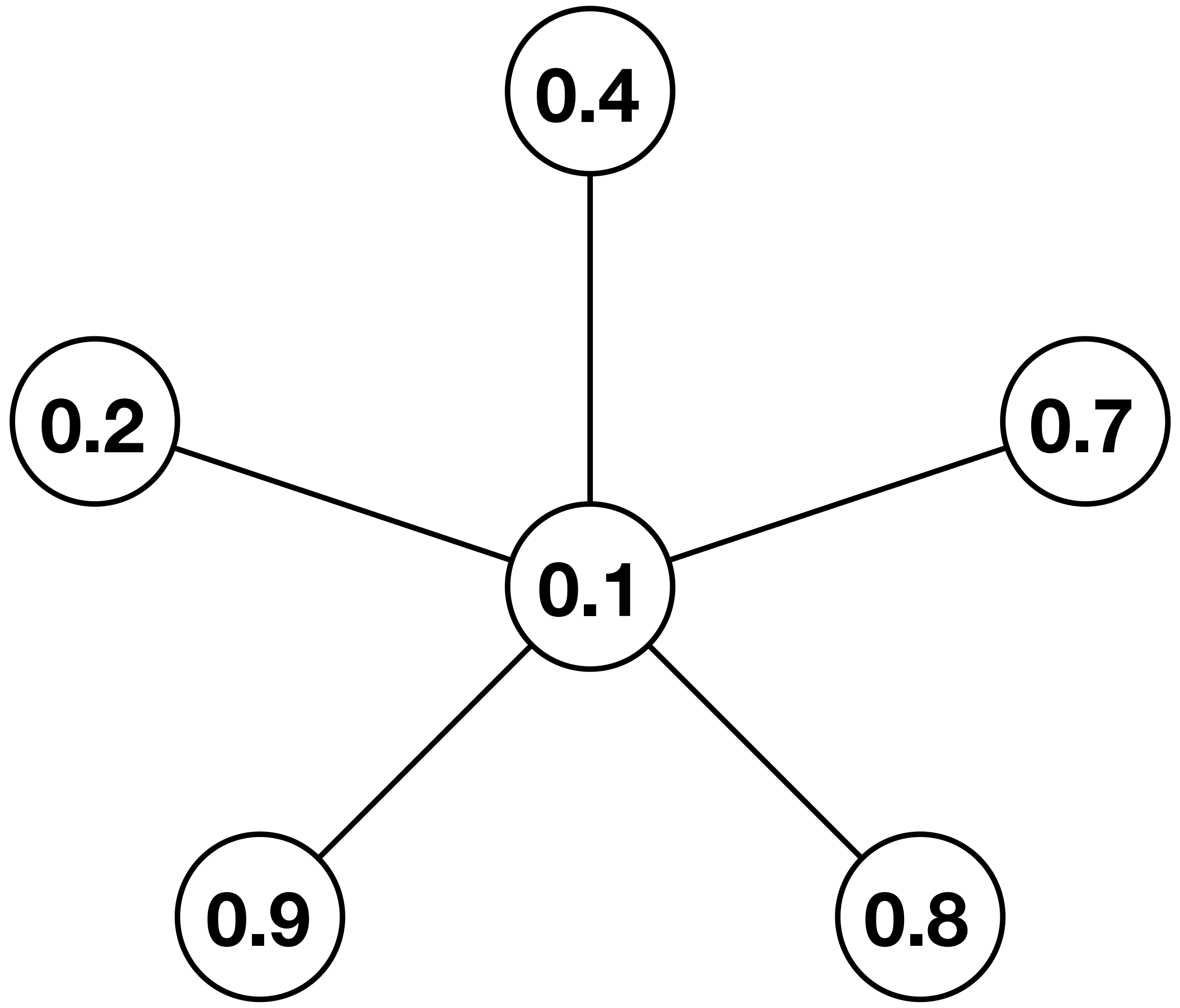

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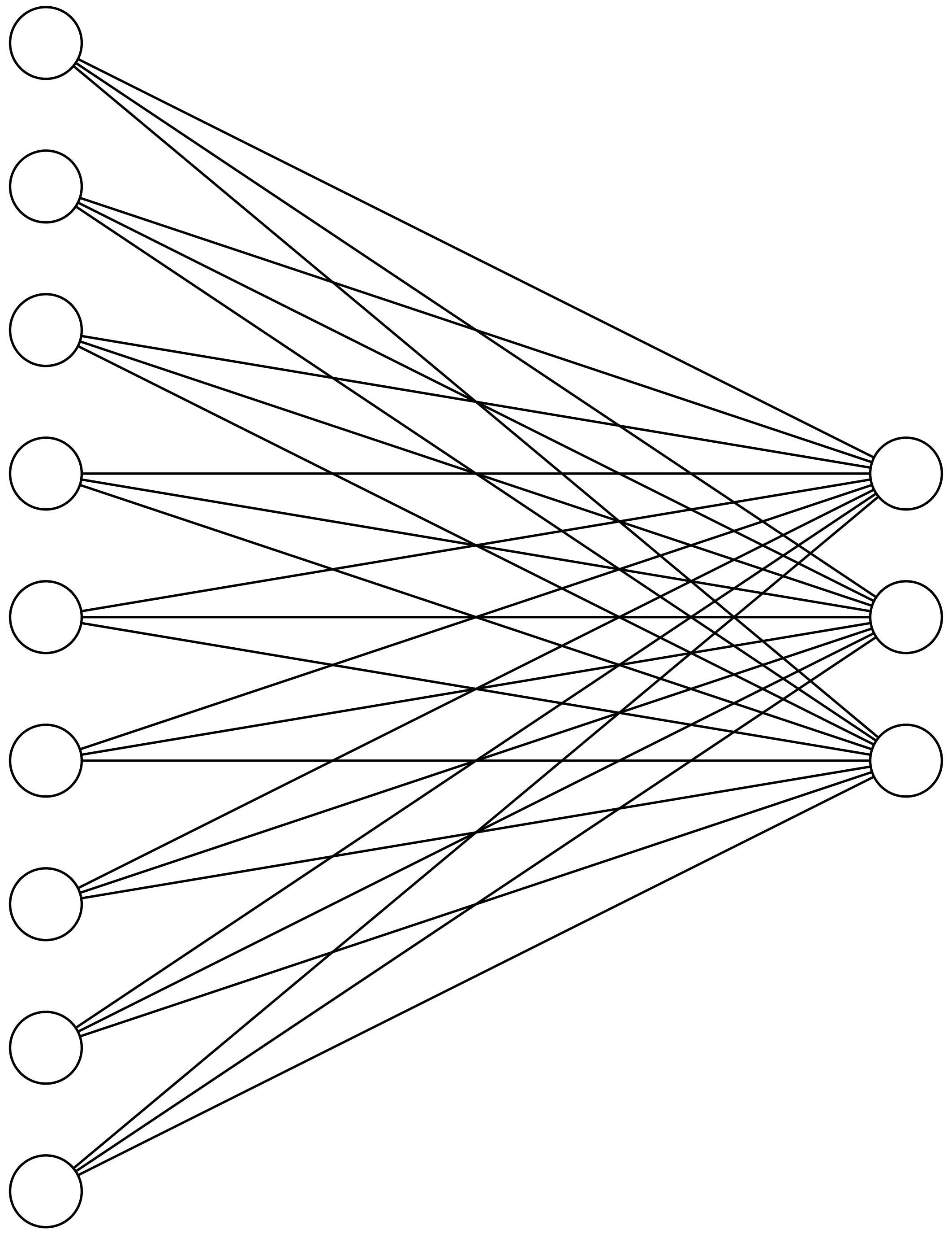

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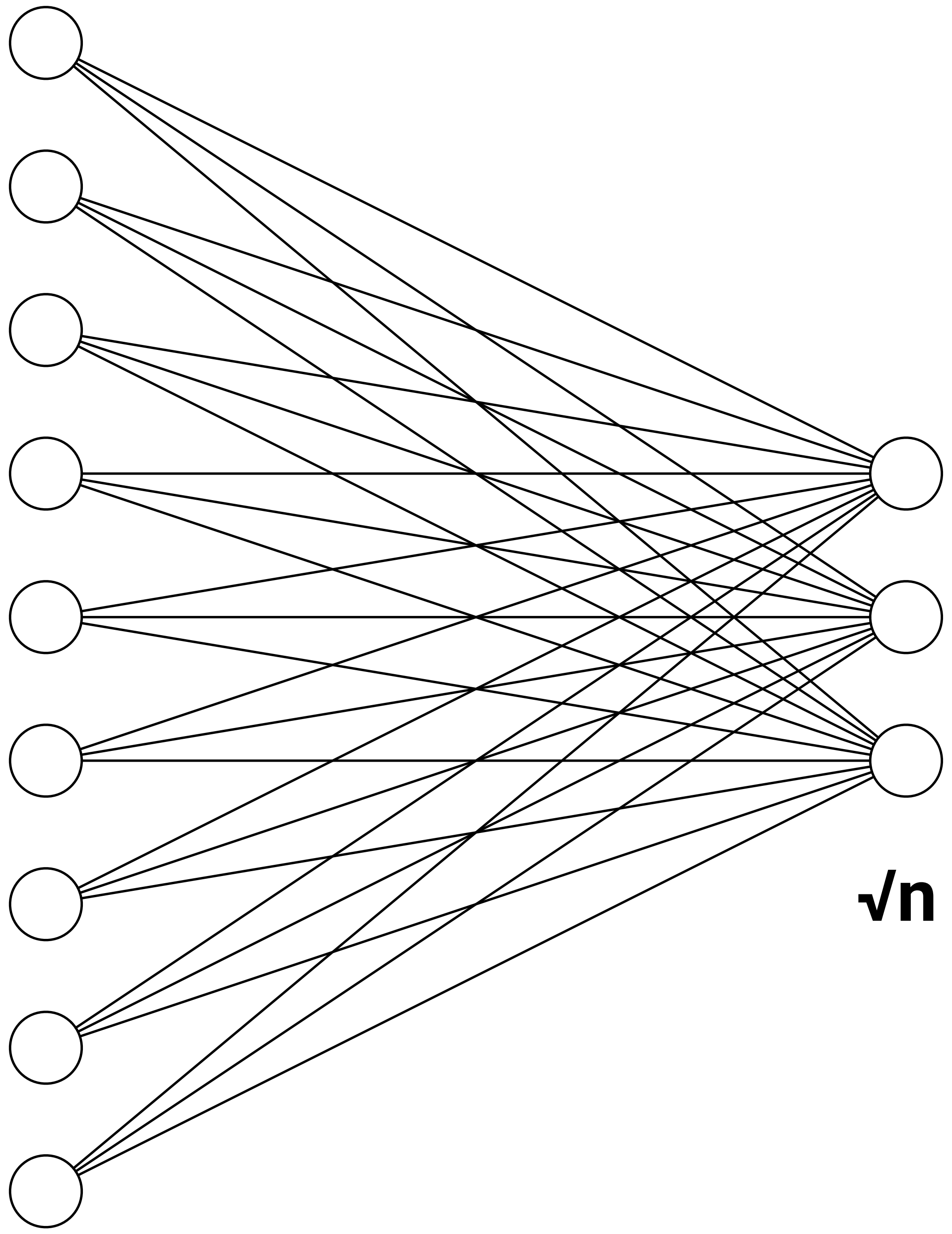

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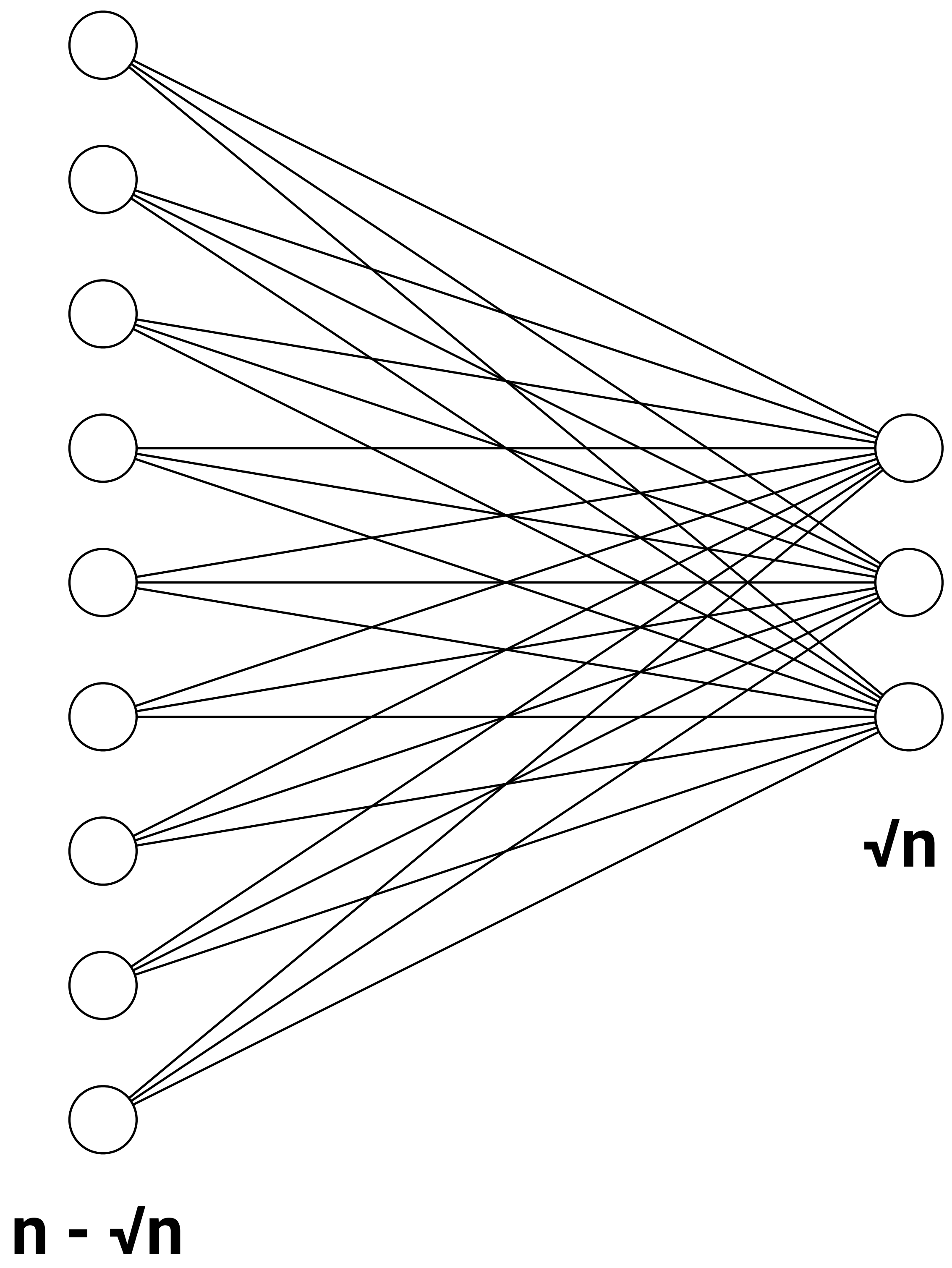

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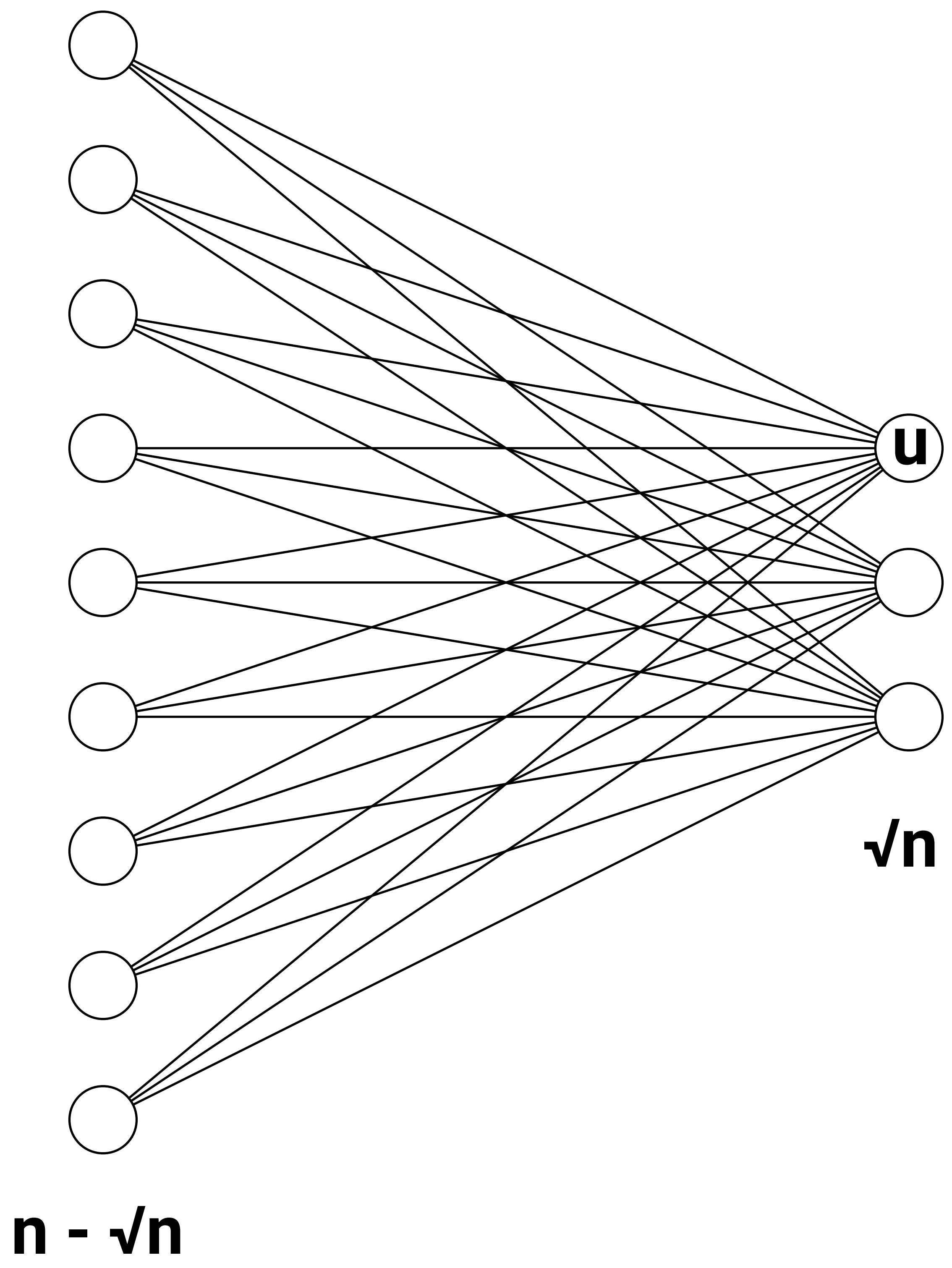

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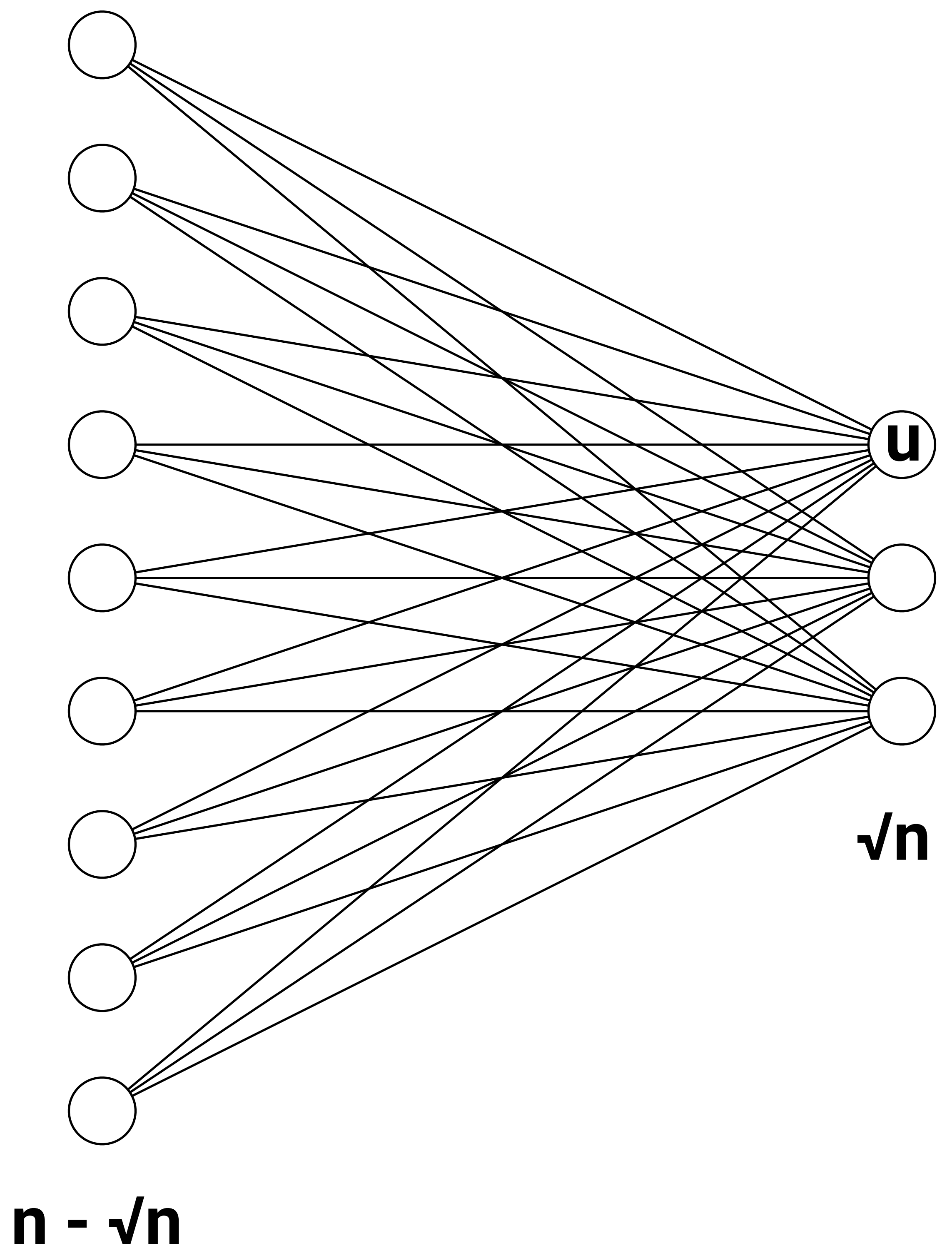

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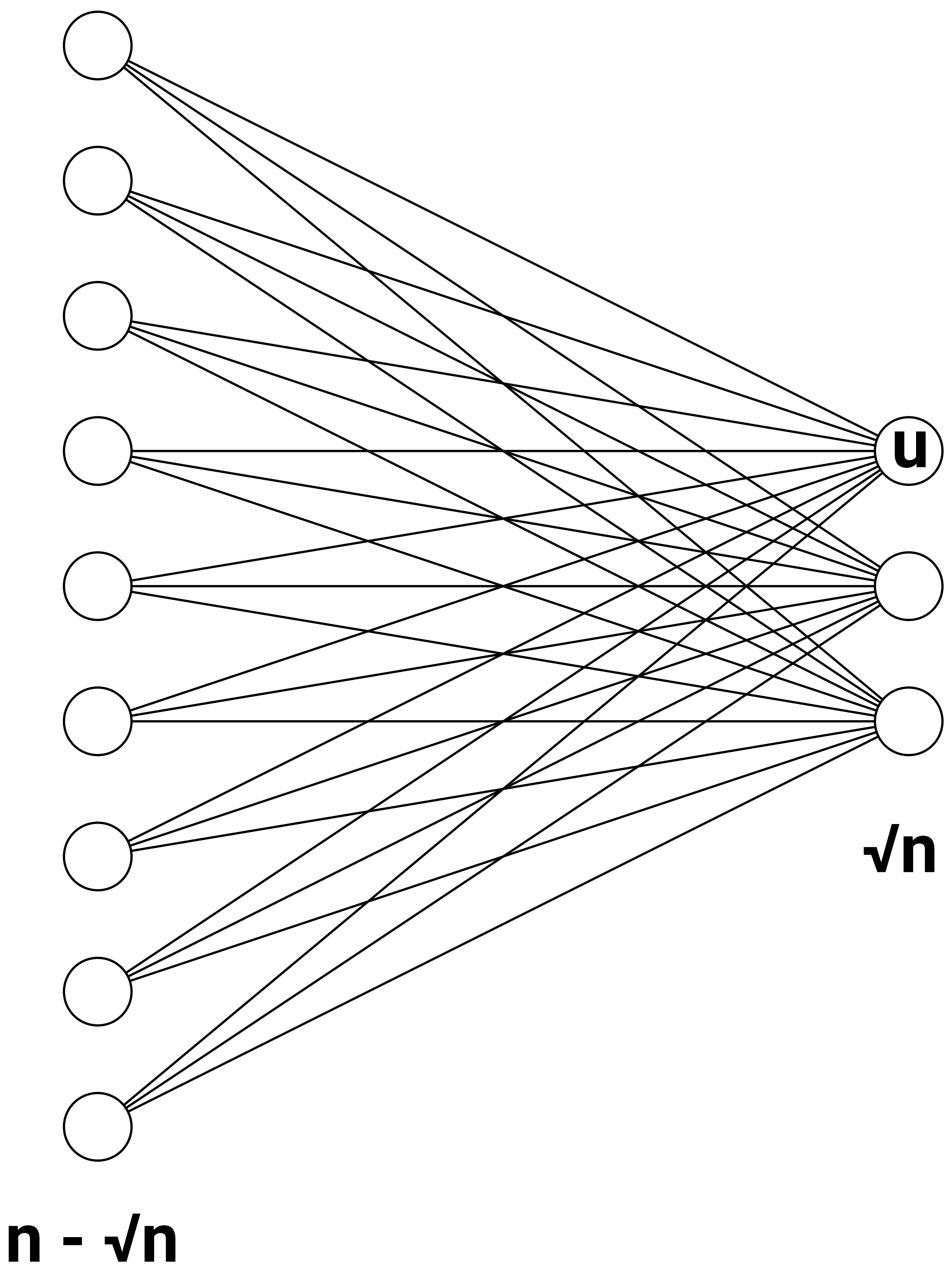

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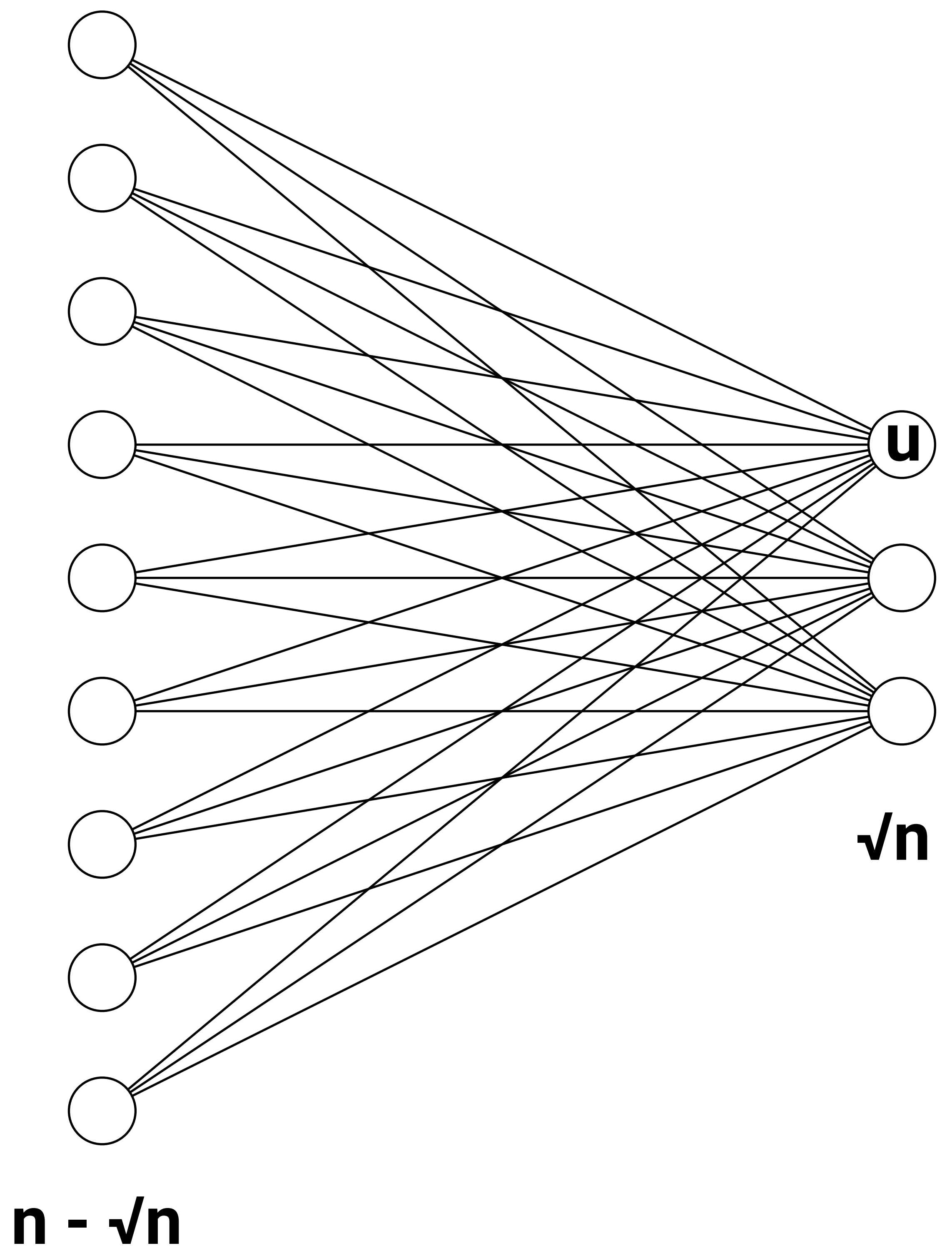

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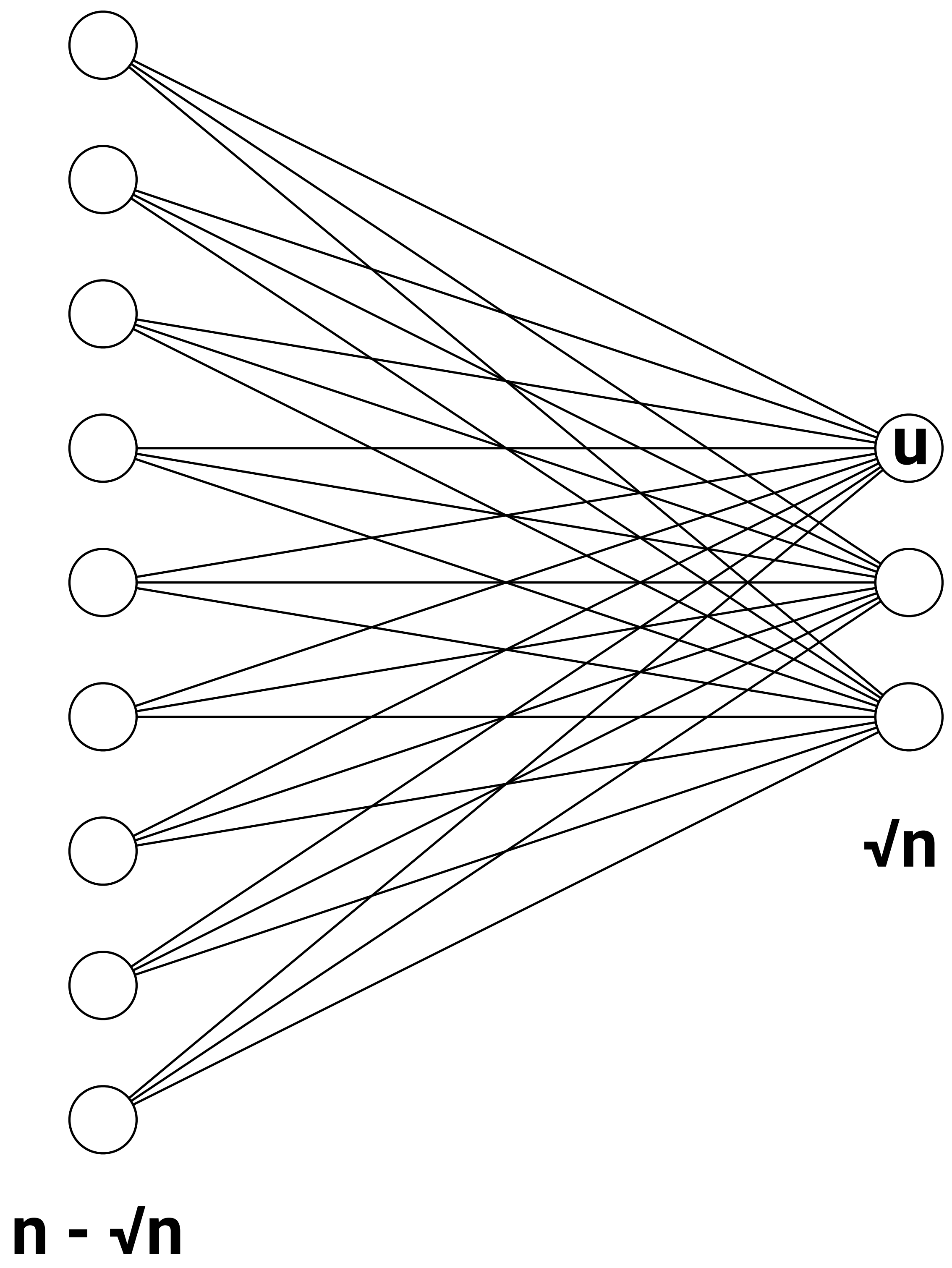

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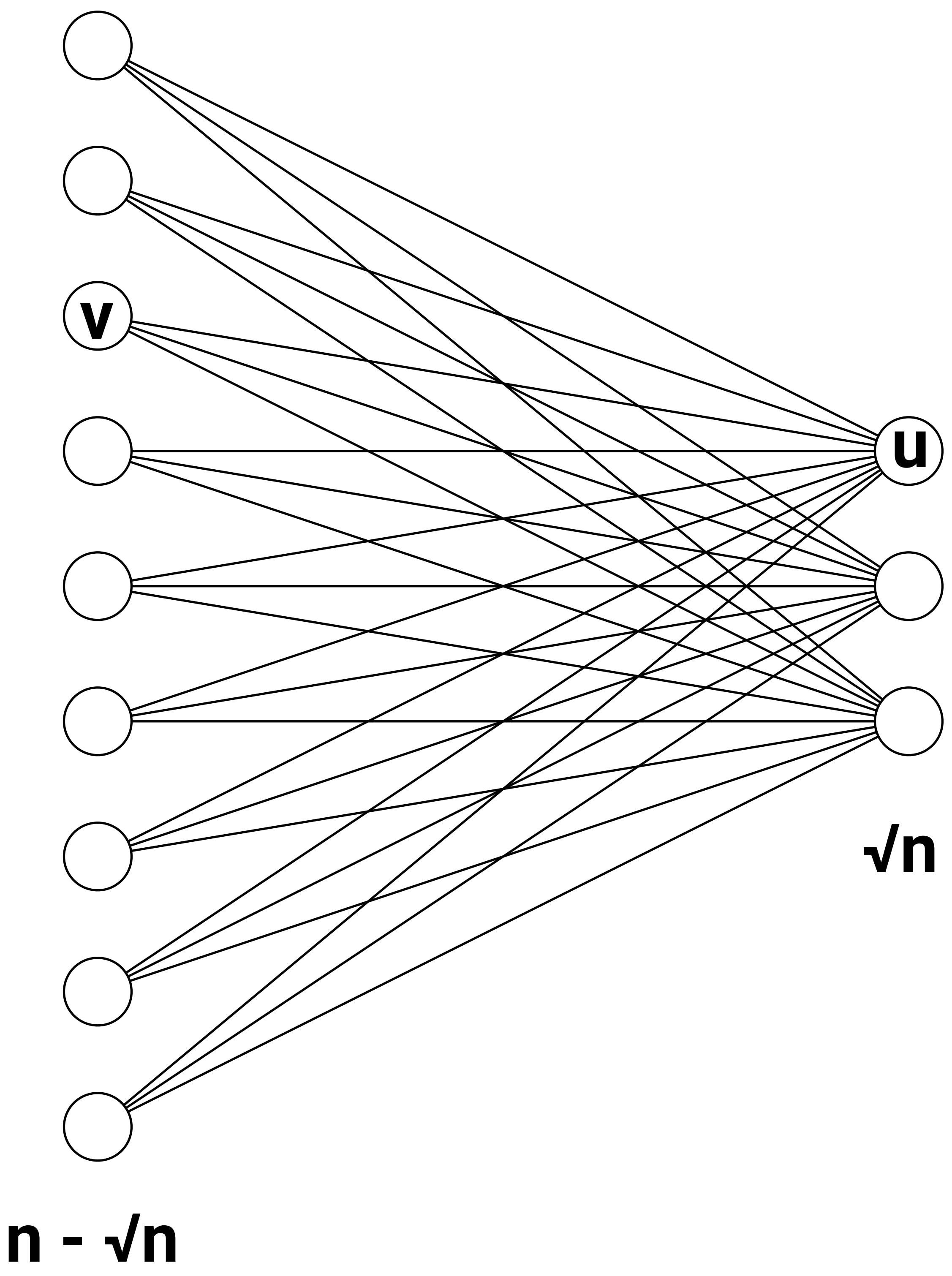

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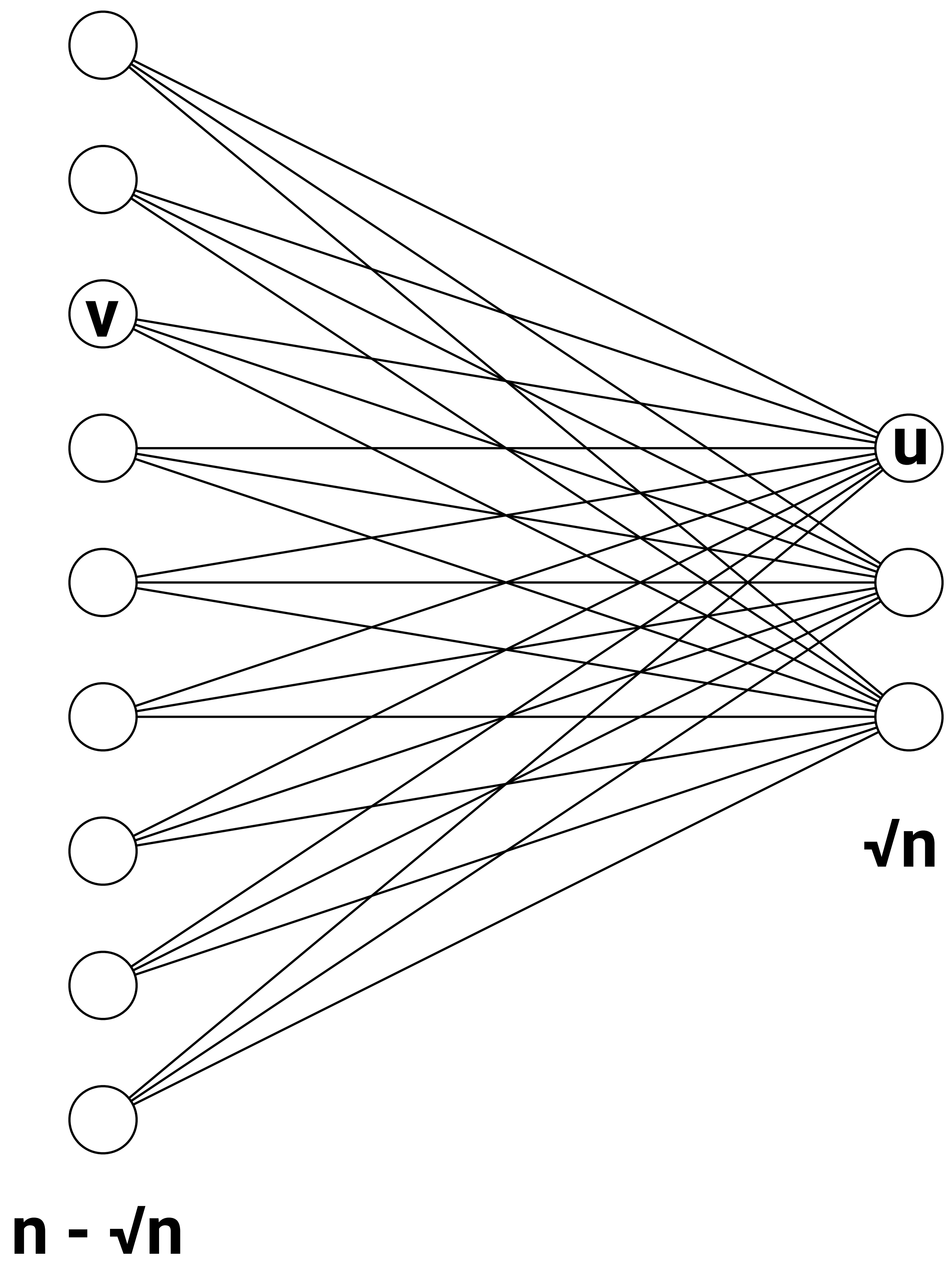

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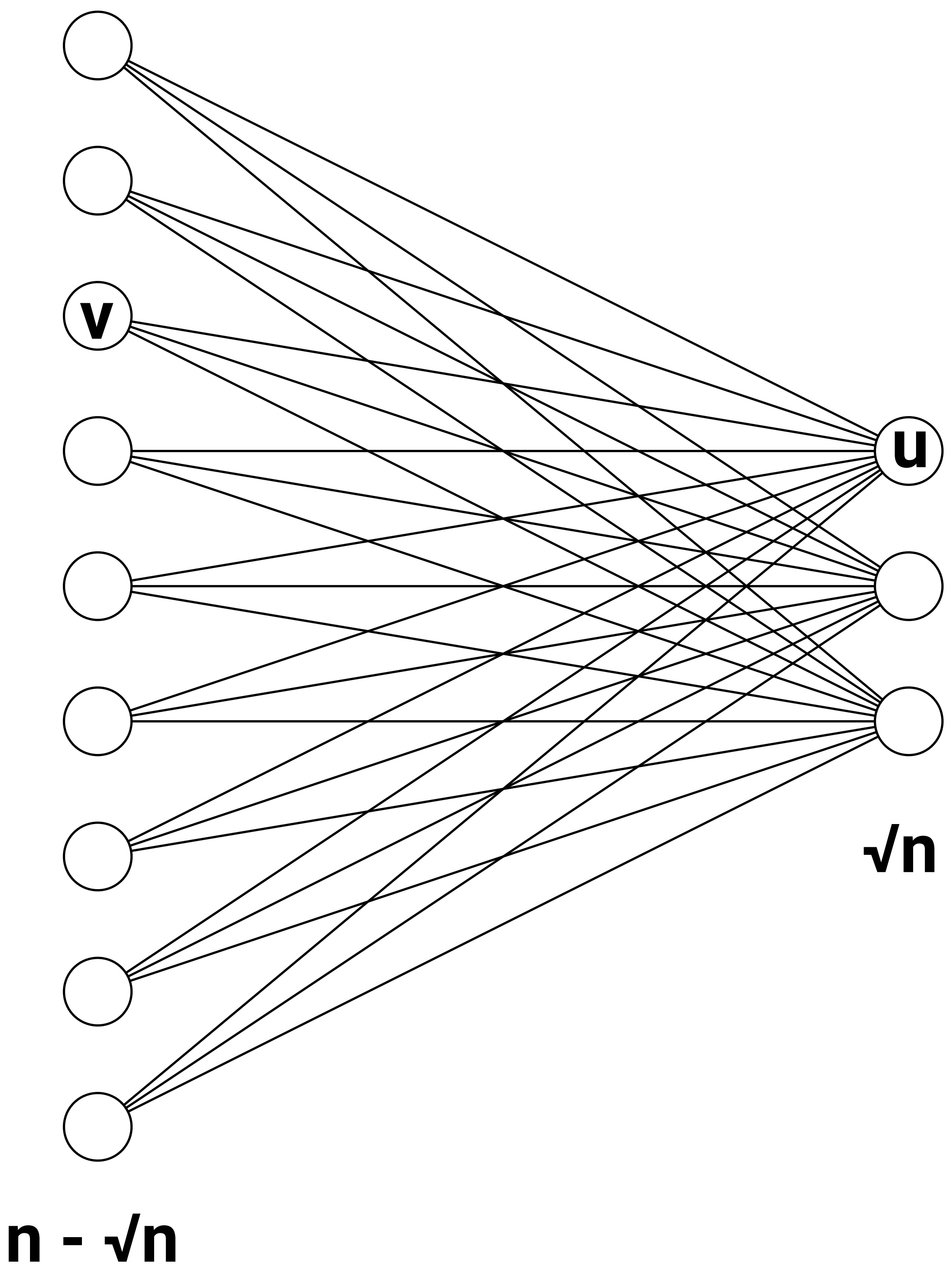

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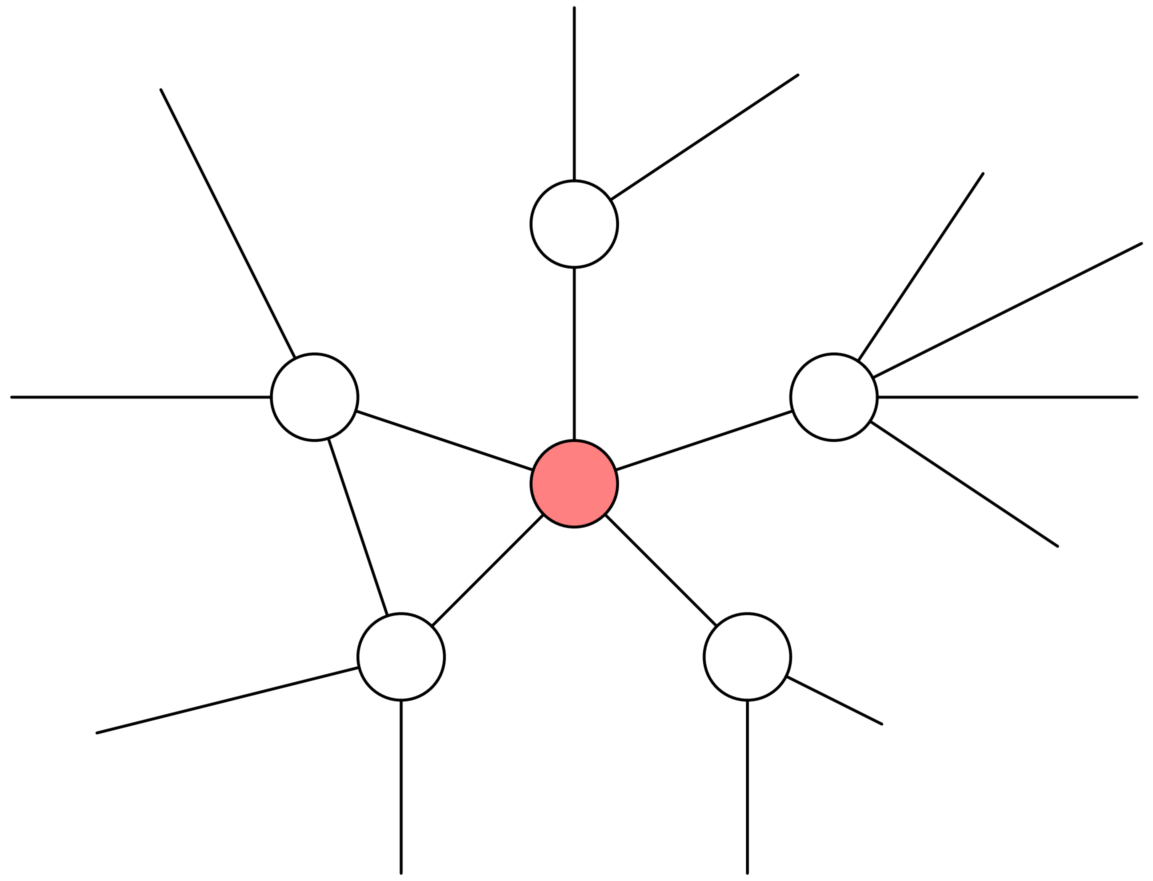

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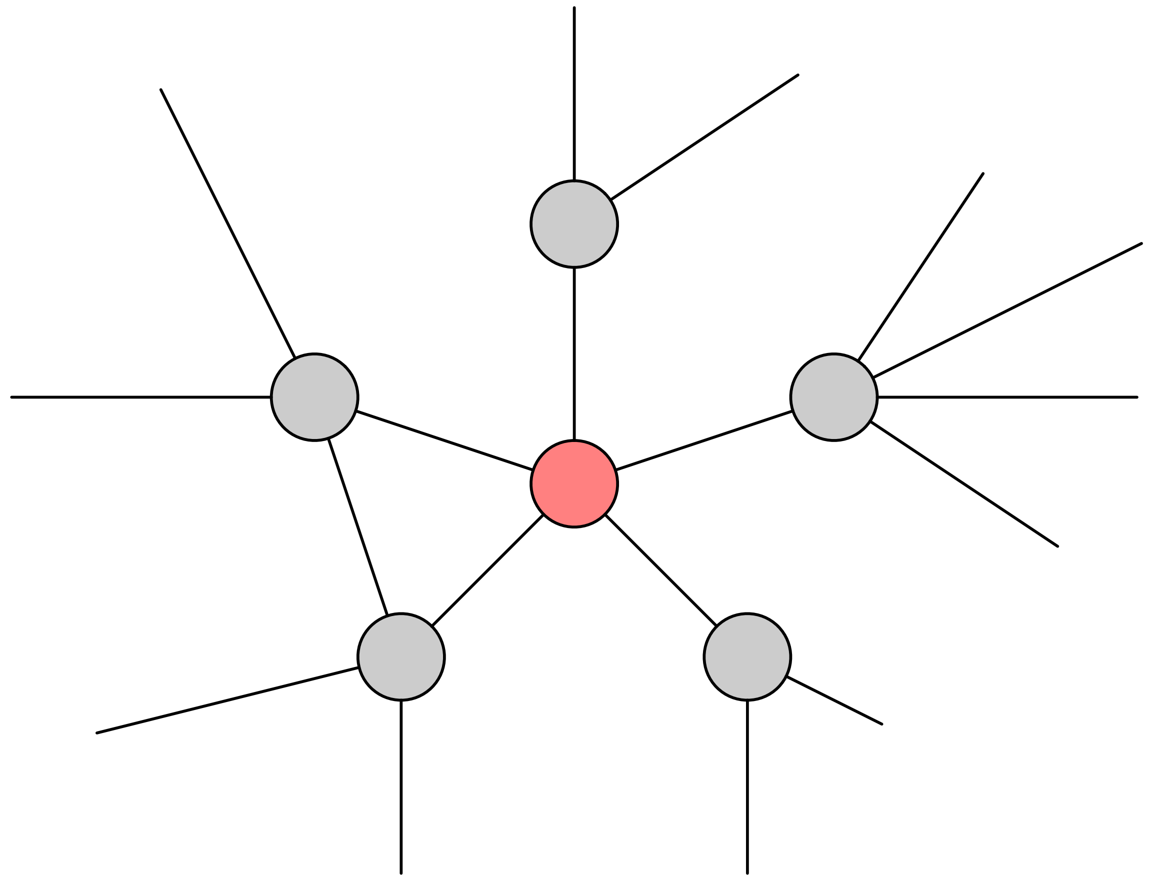

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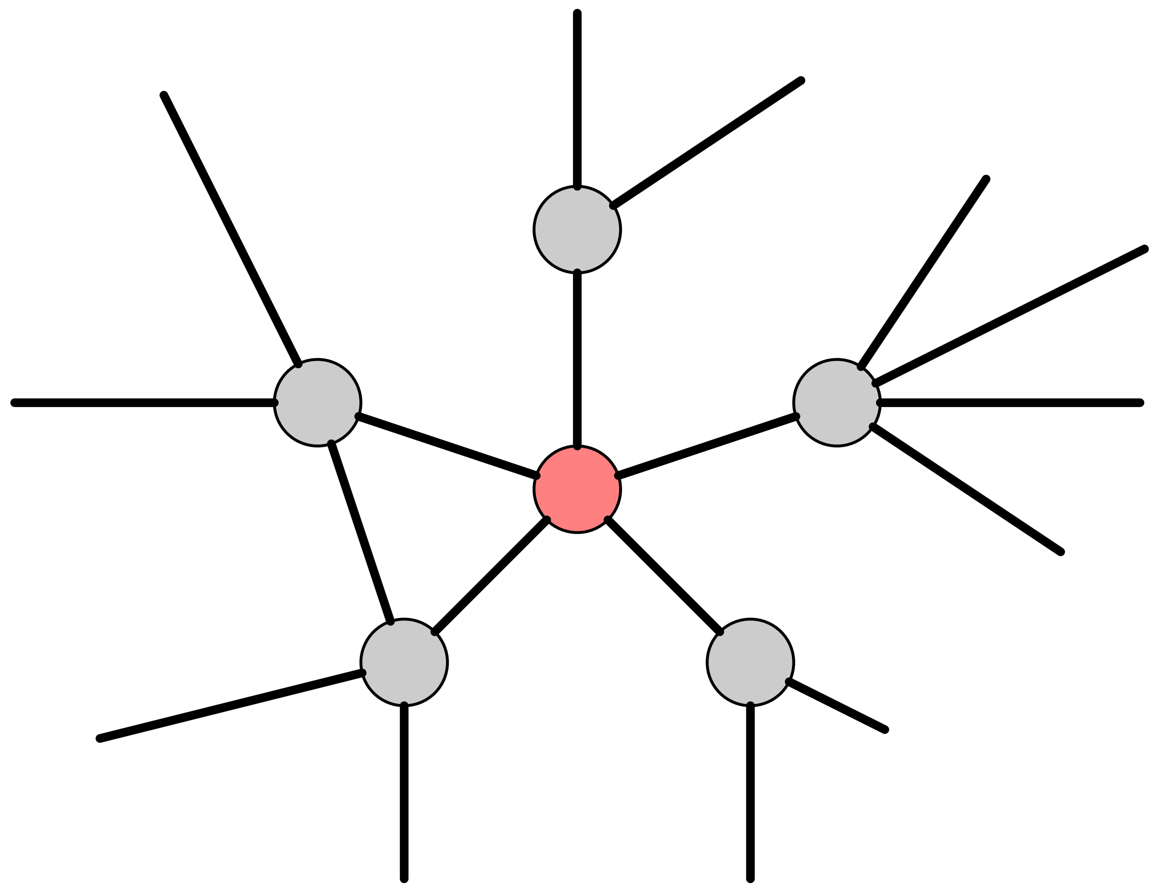

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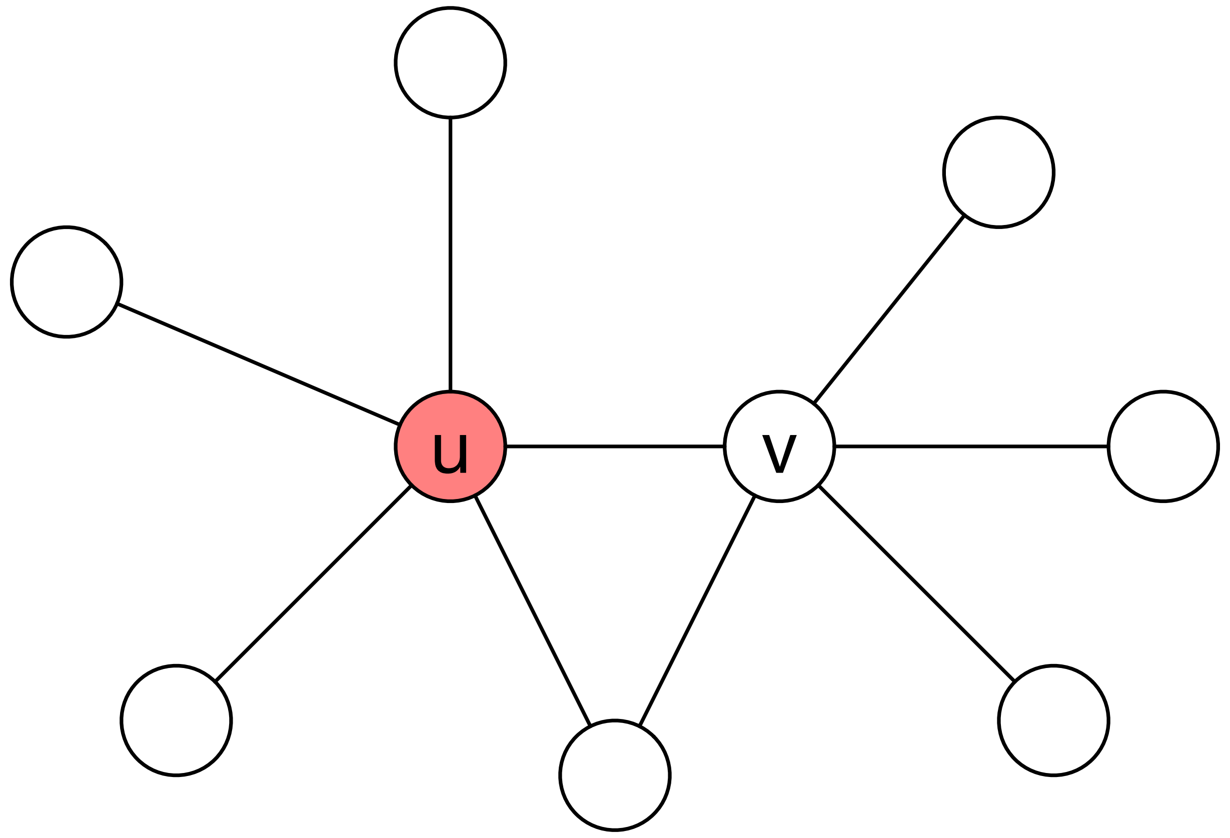

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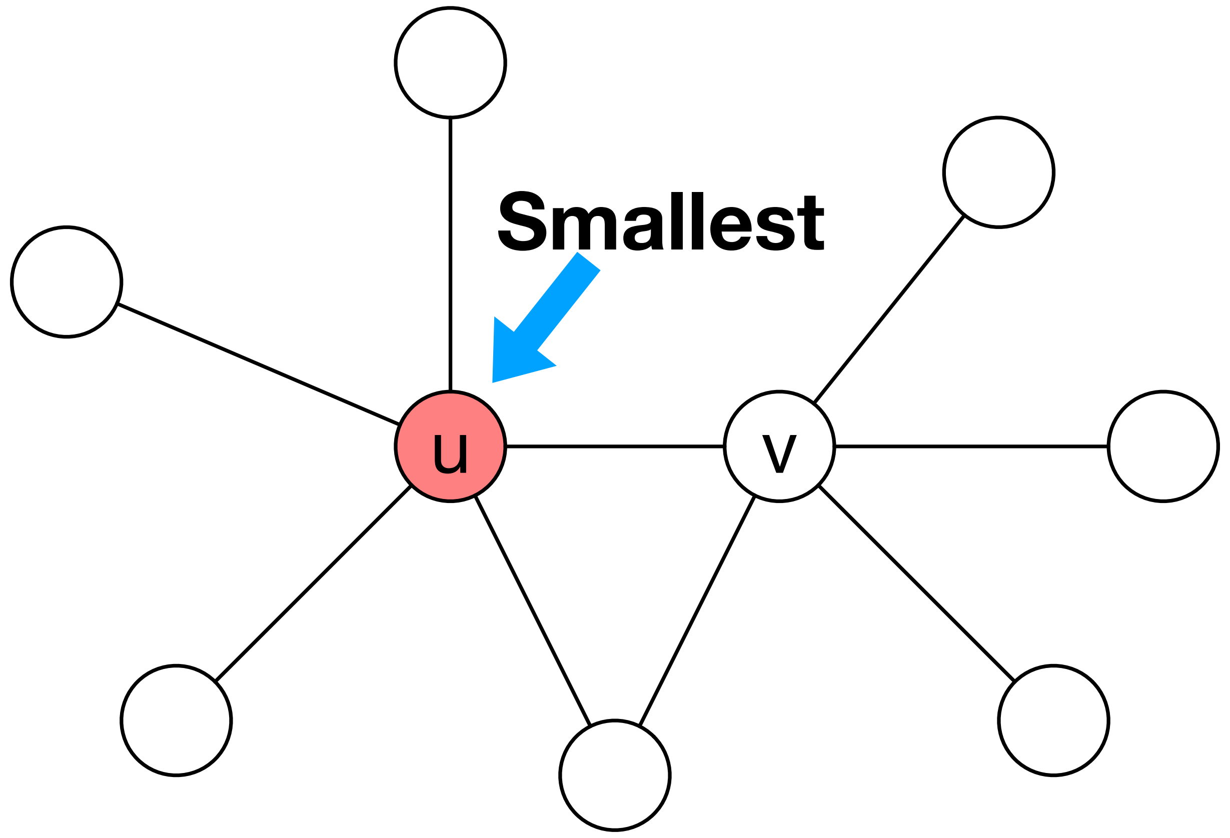

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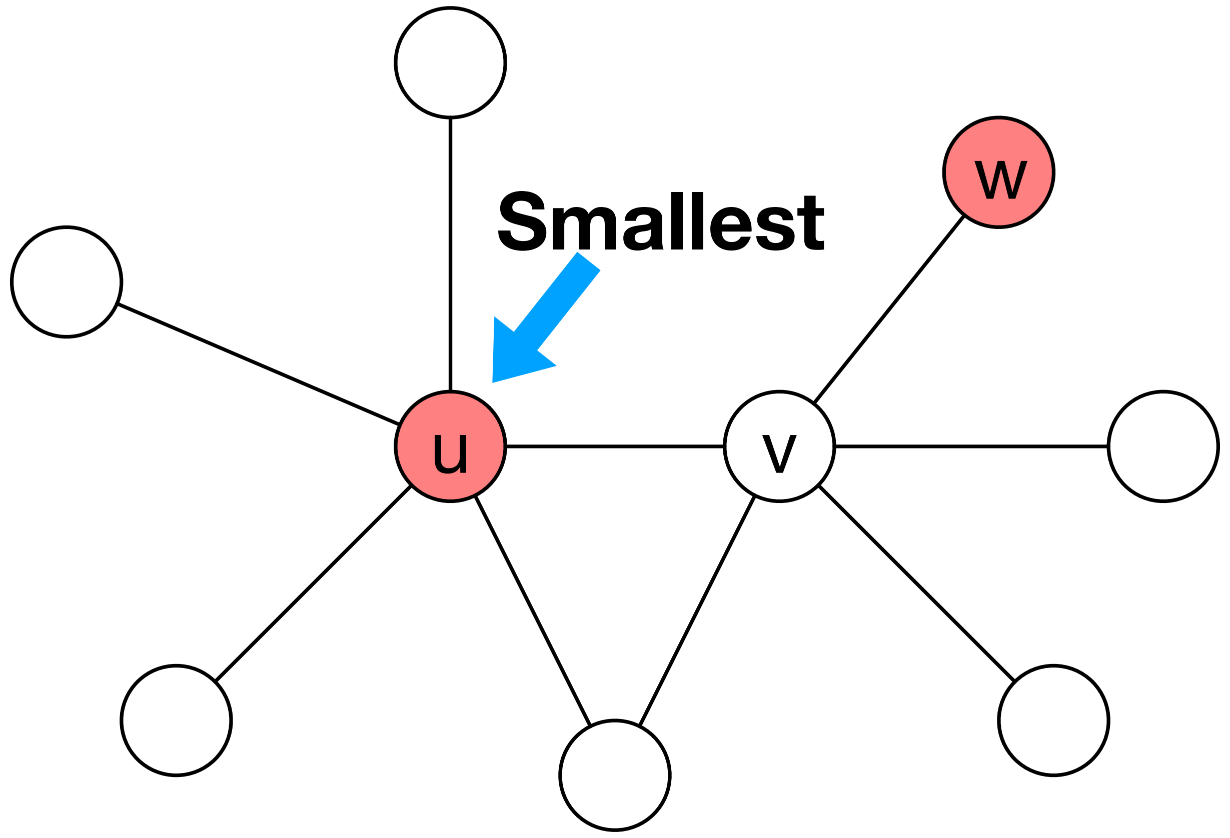

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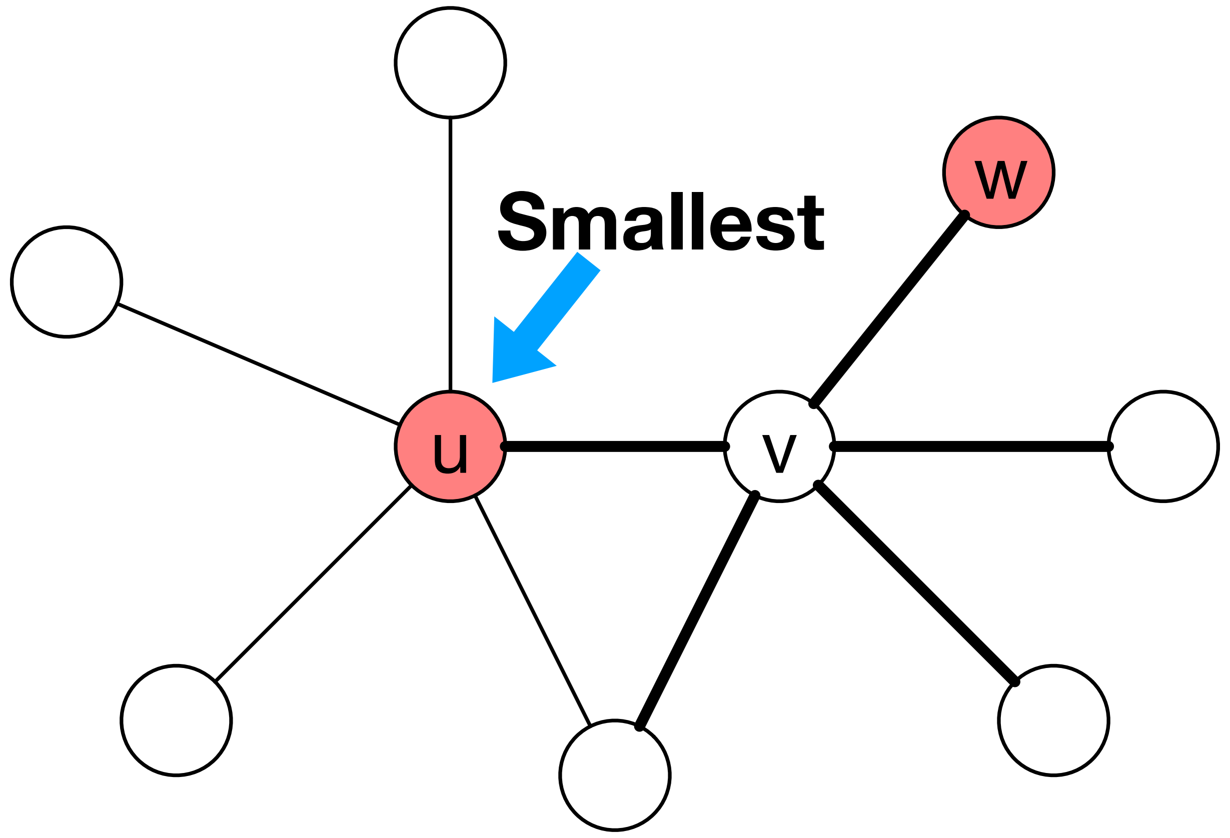

$\mathcal{E}_{u,v}$ is **true** ⇒ all edges of **v** get removed "**because of u**"

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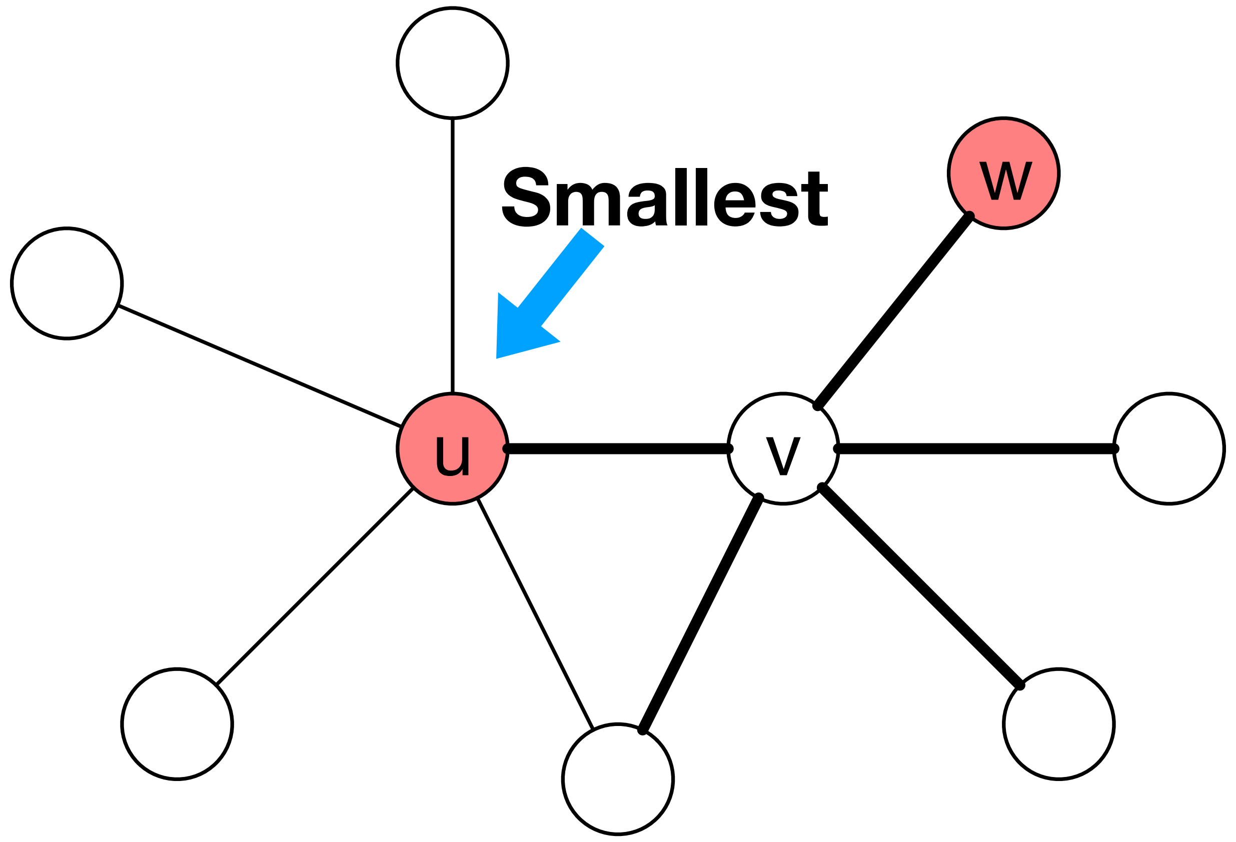

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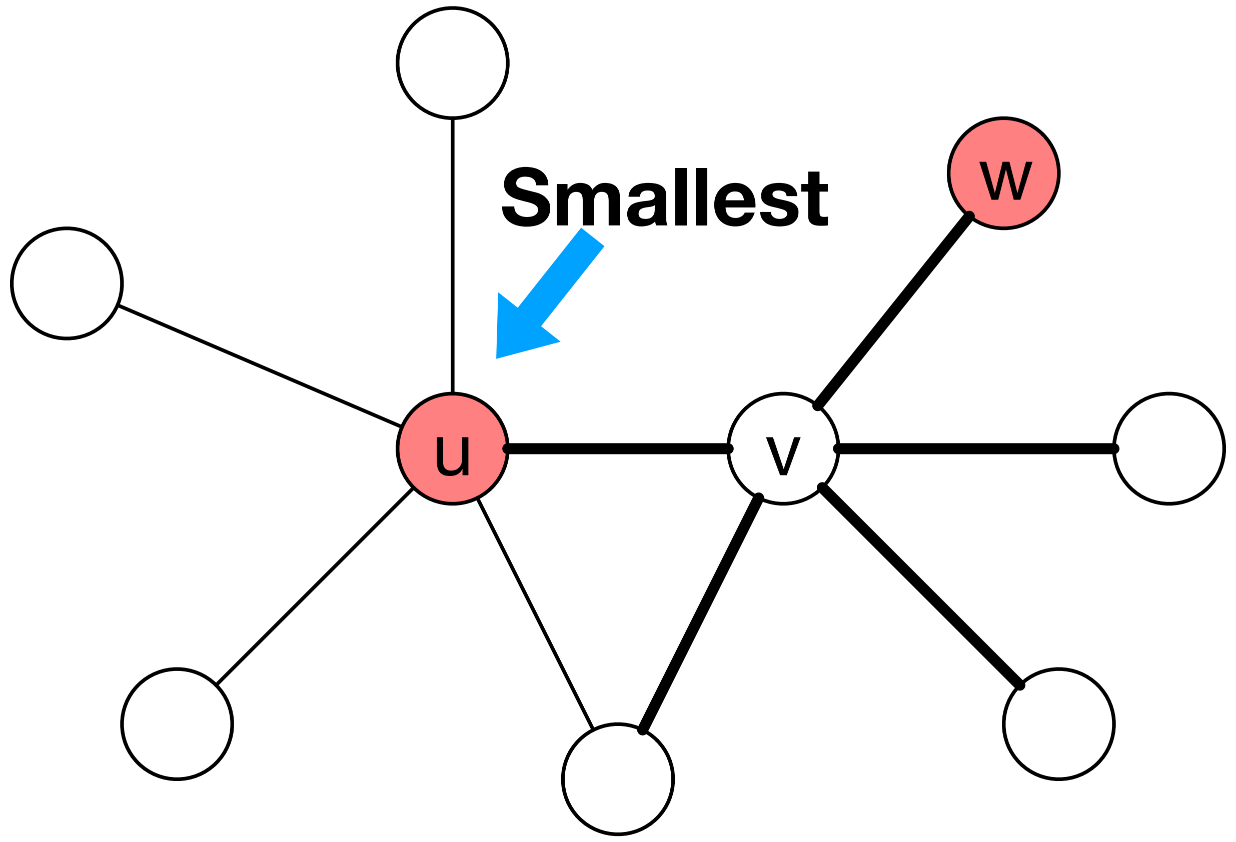

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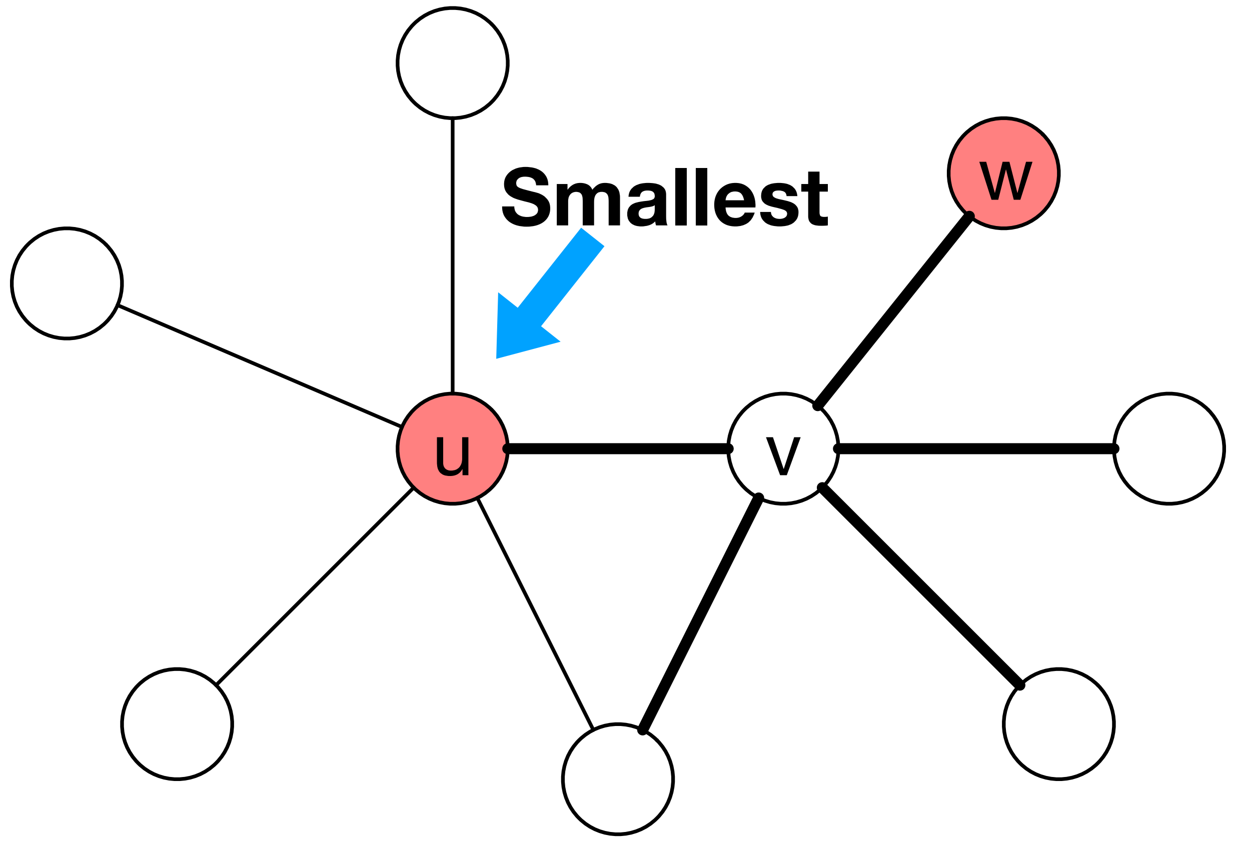

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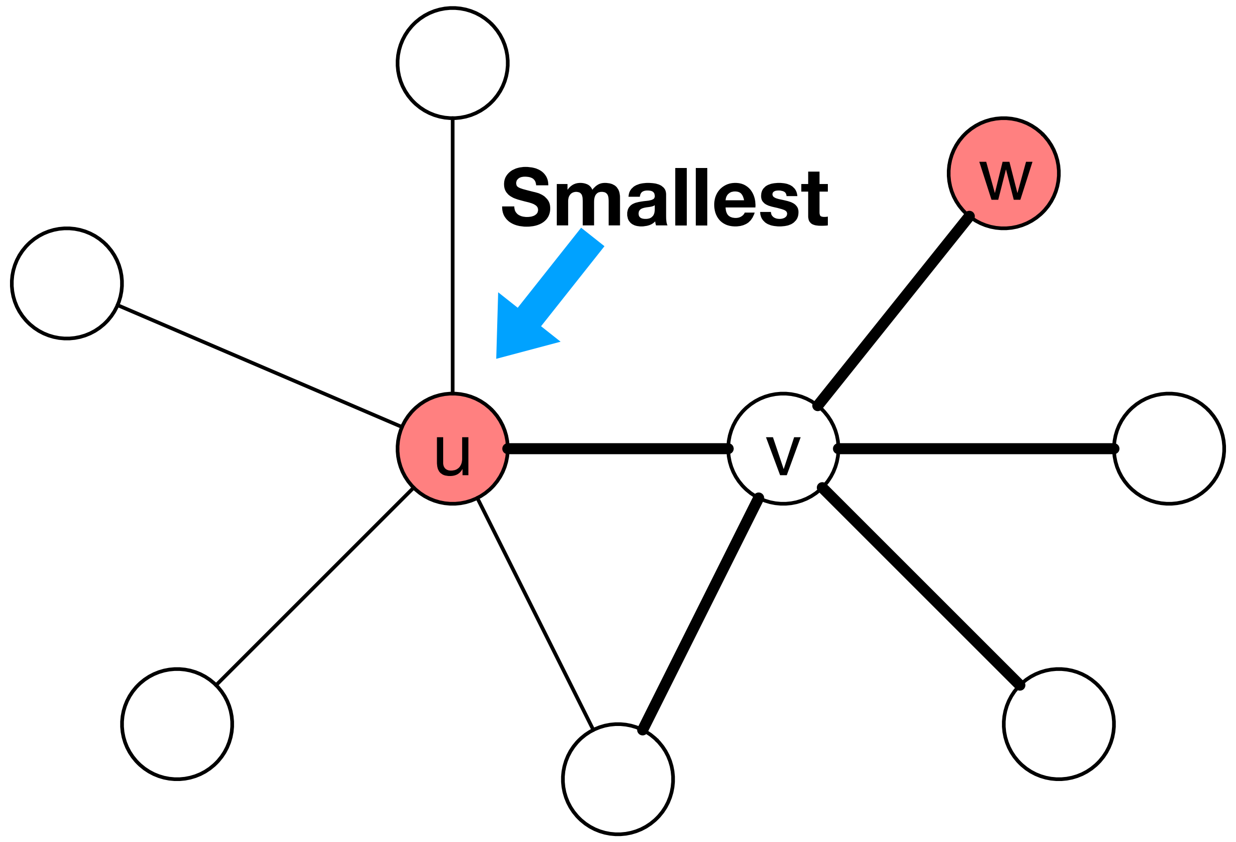

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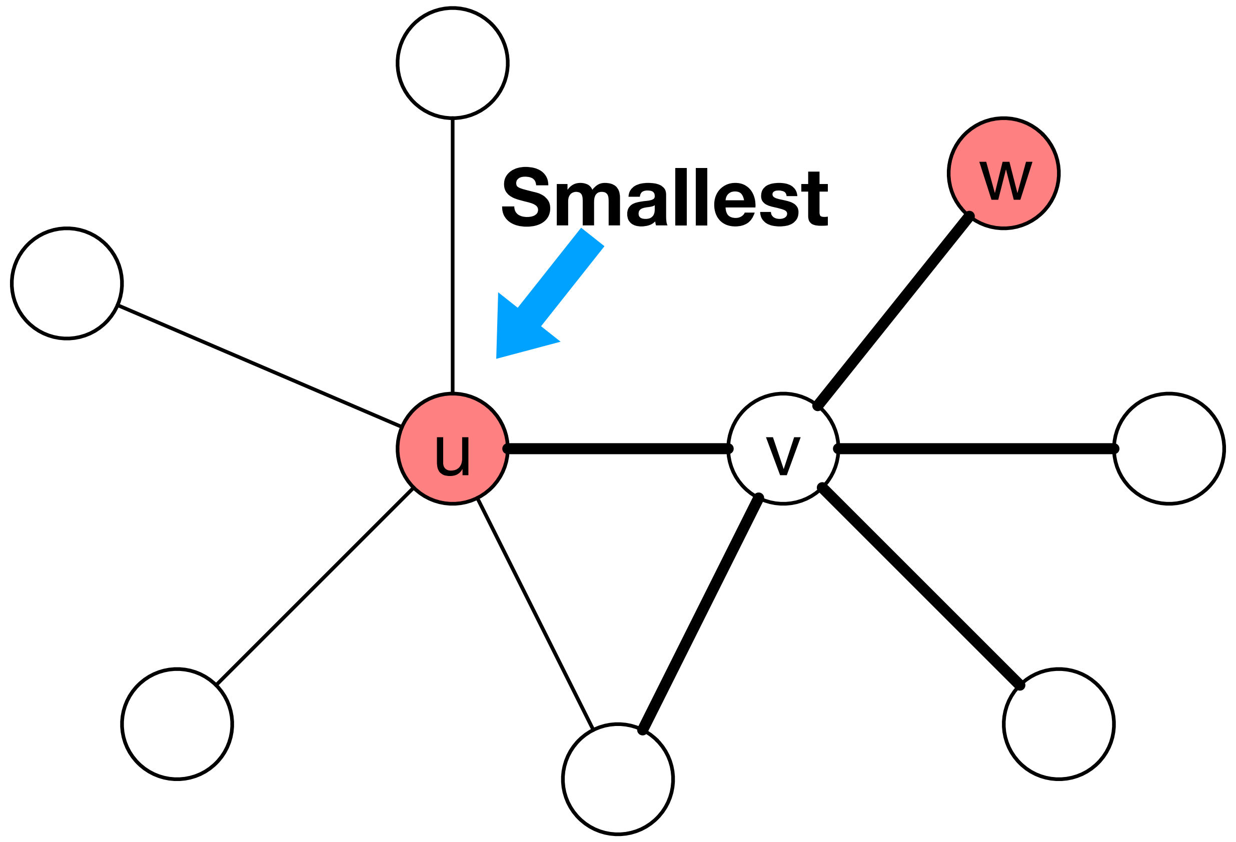

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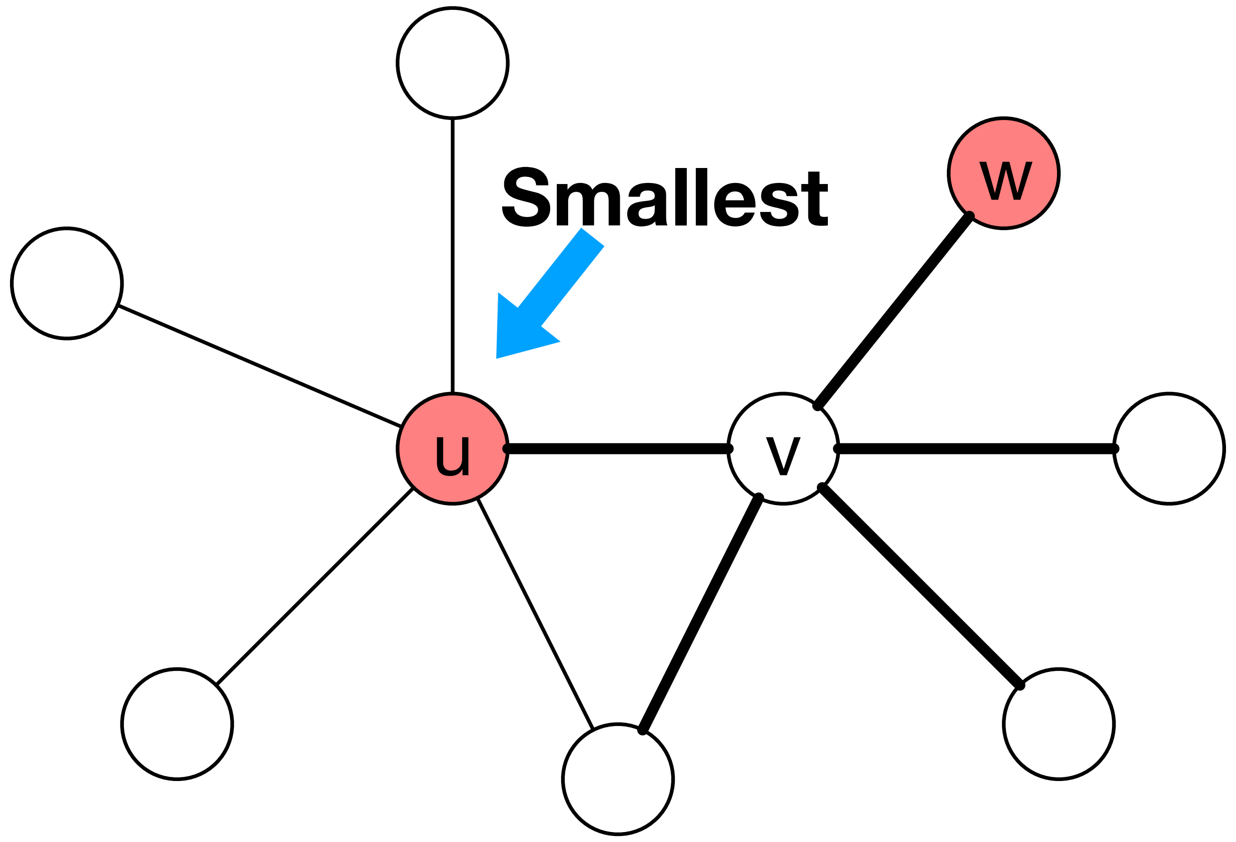

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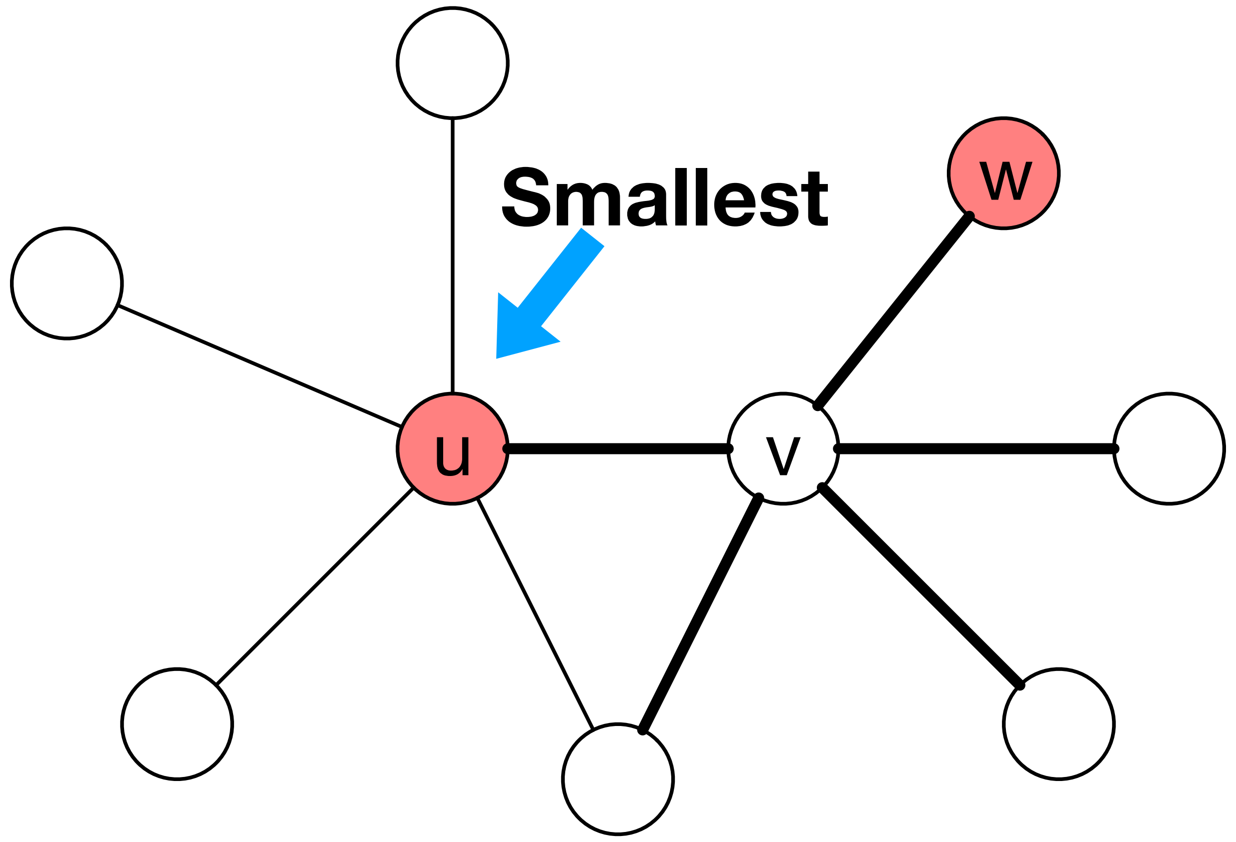

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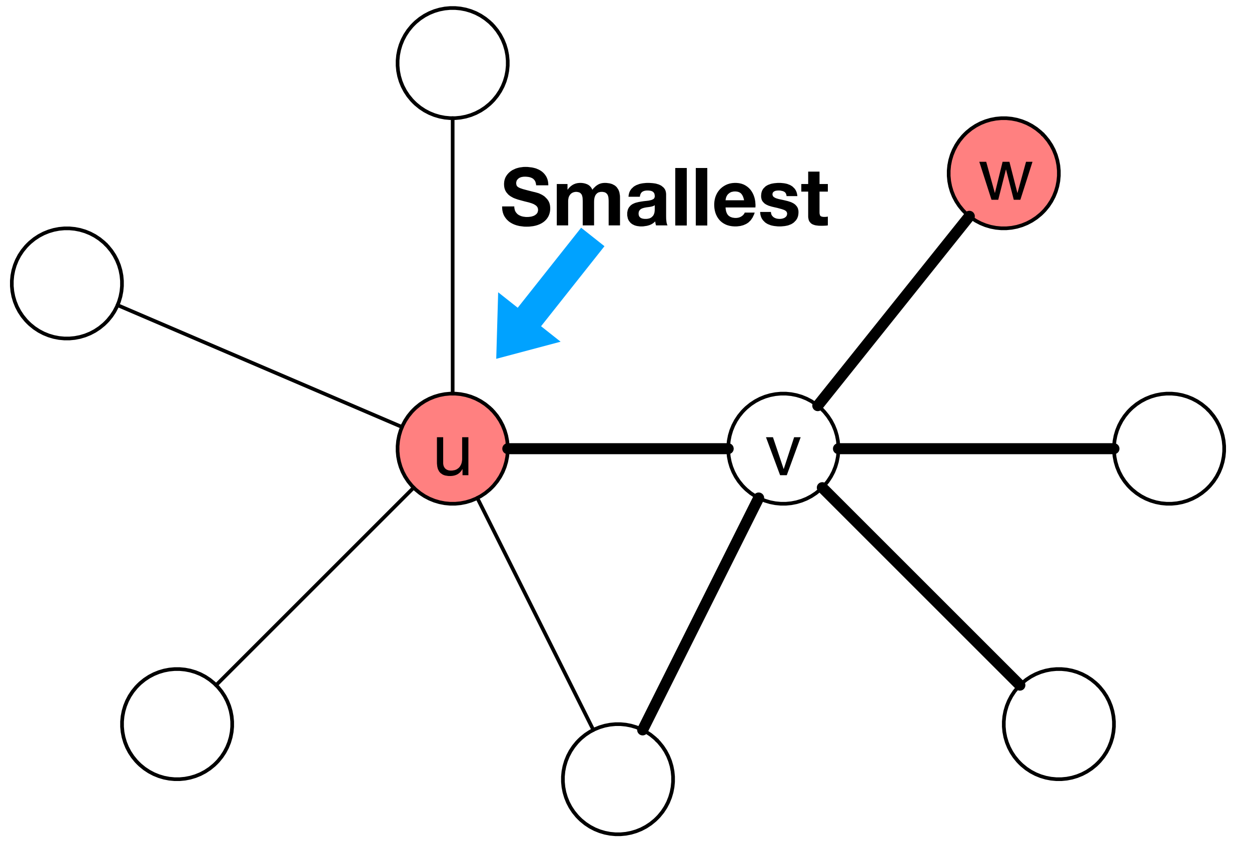

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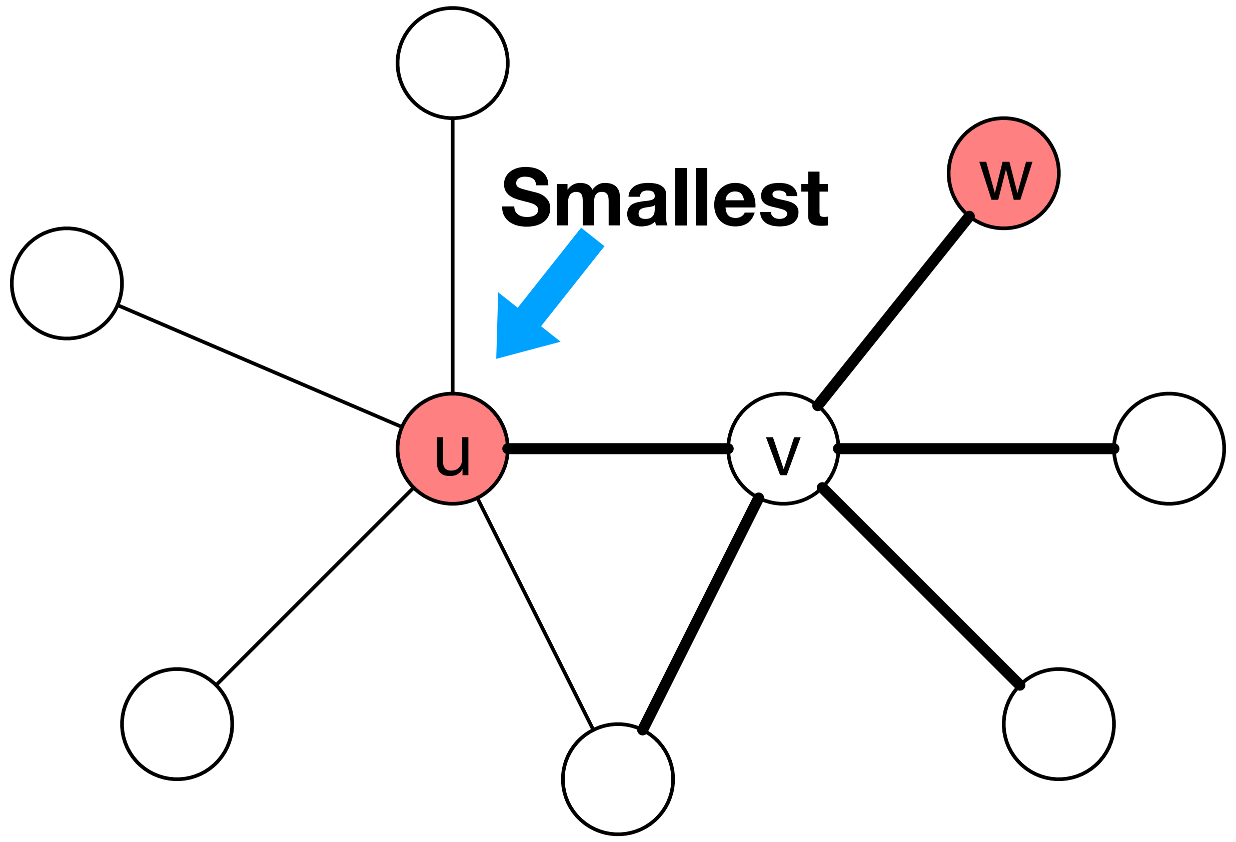

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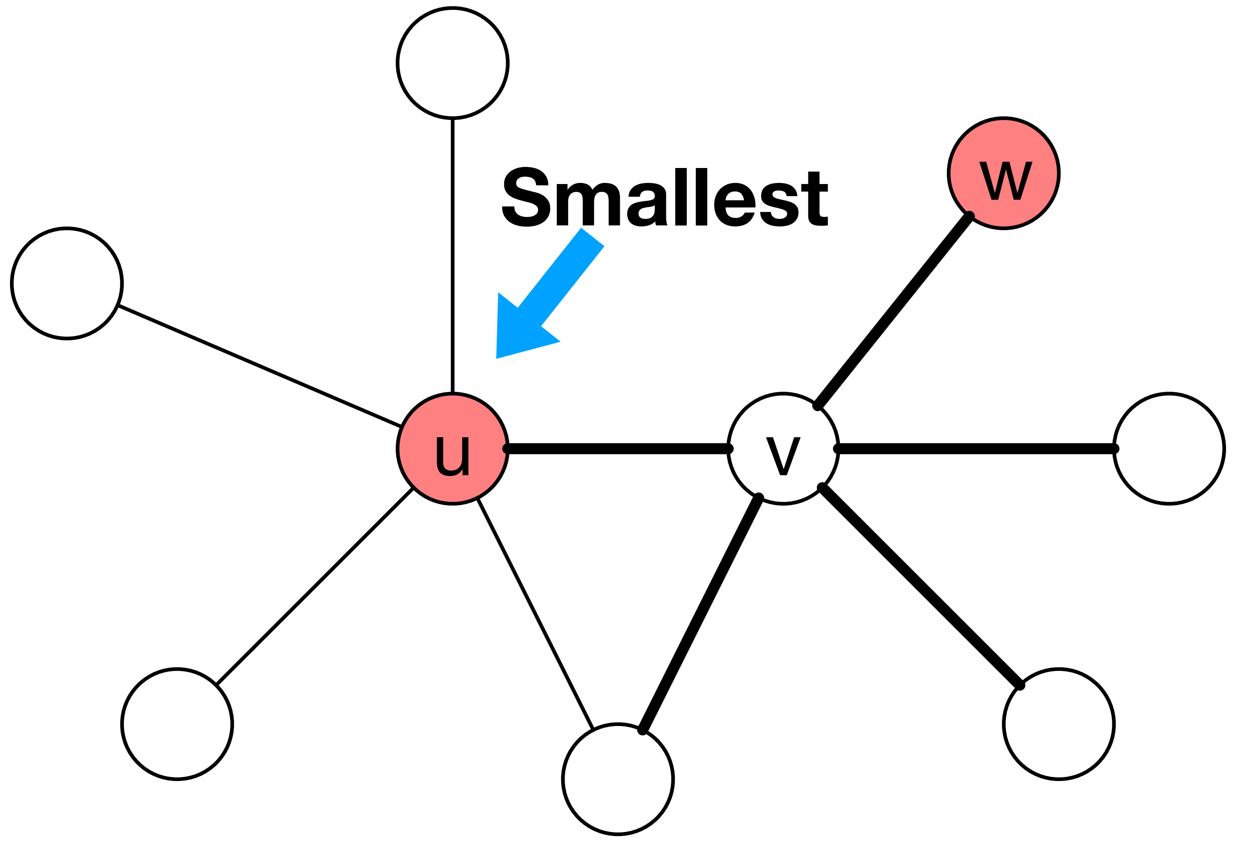

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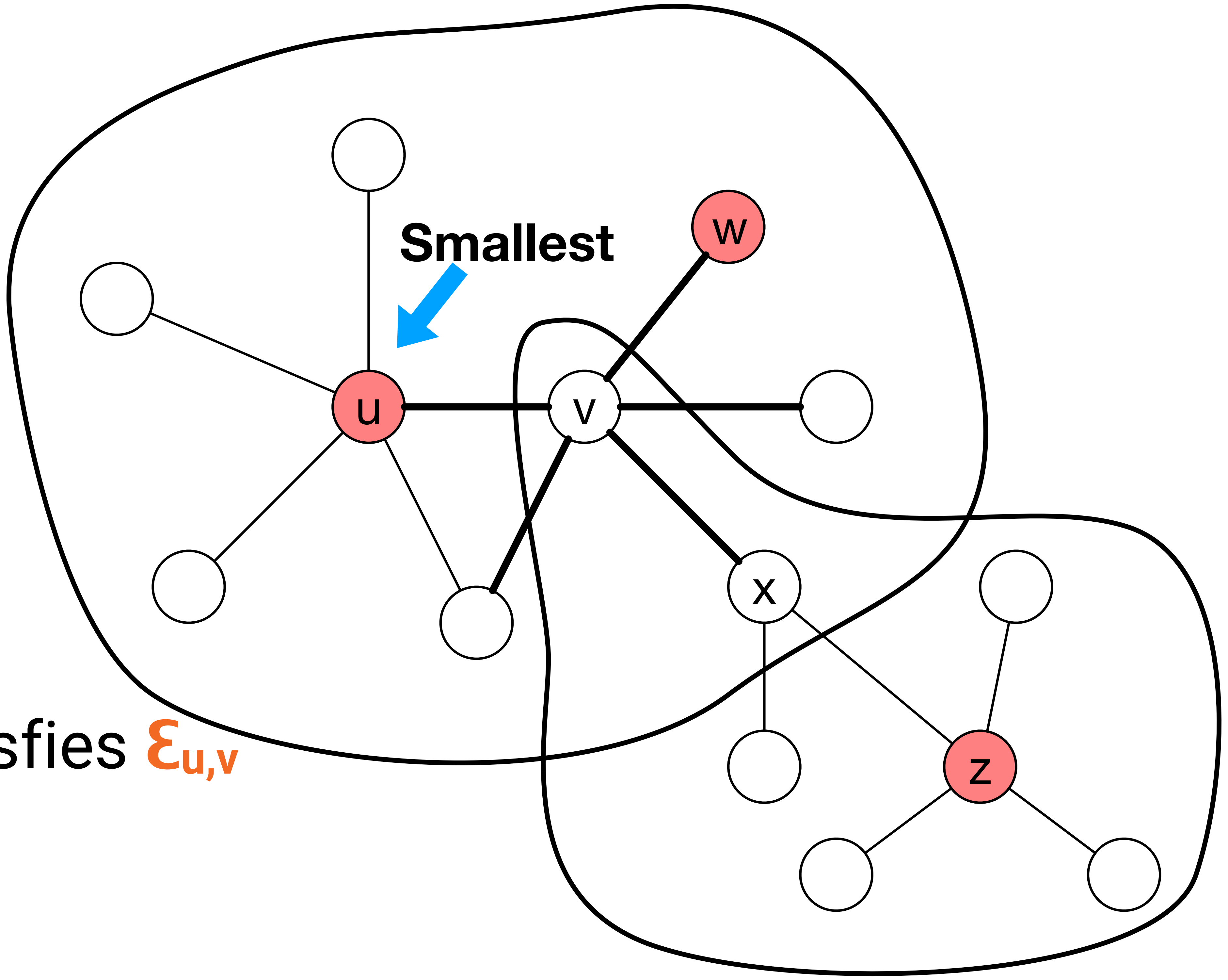

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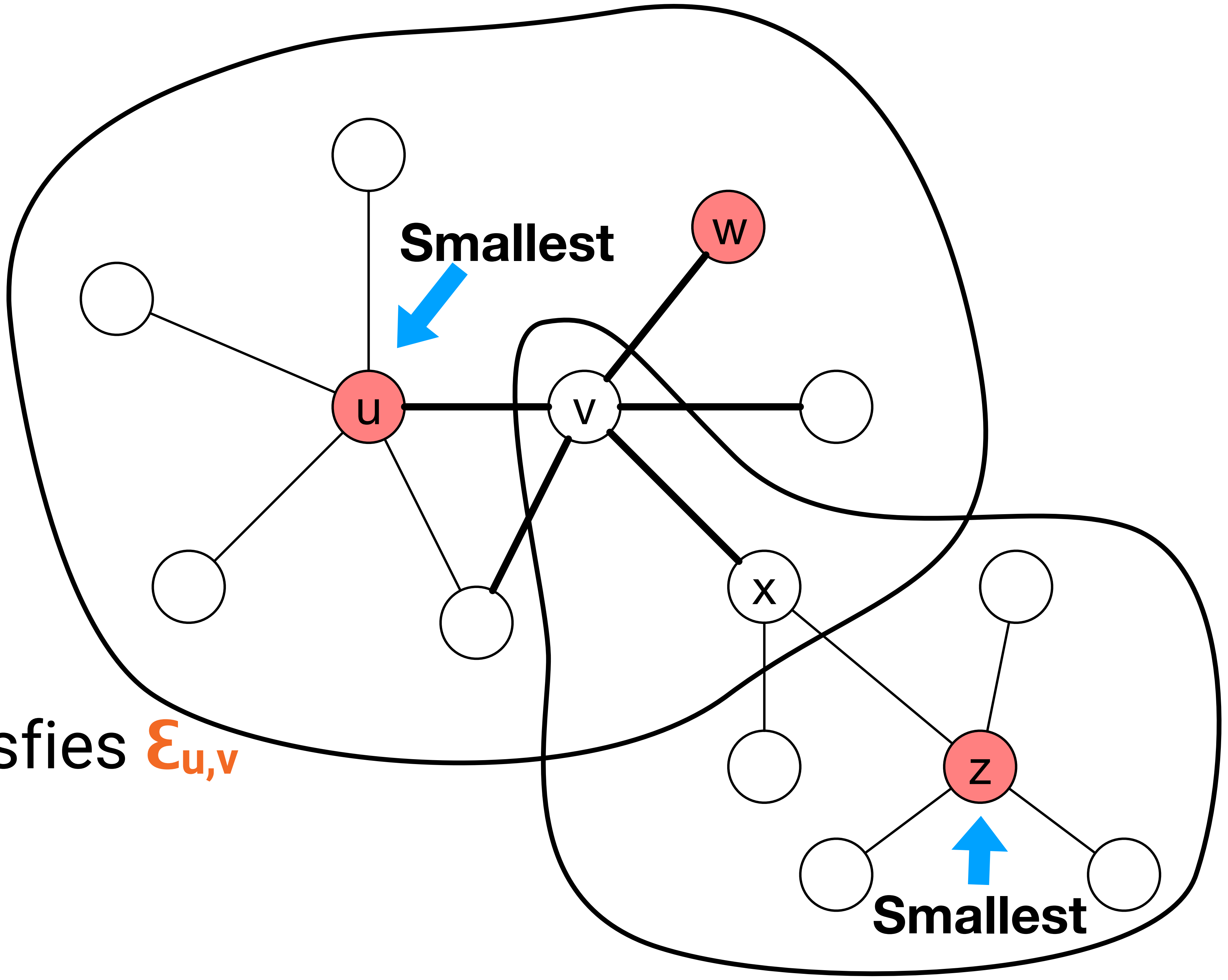

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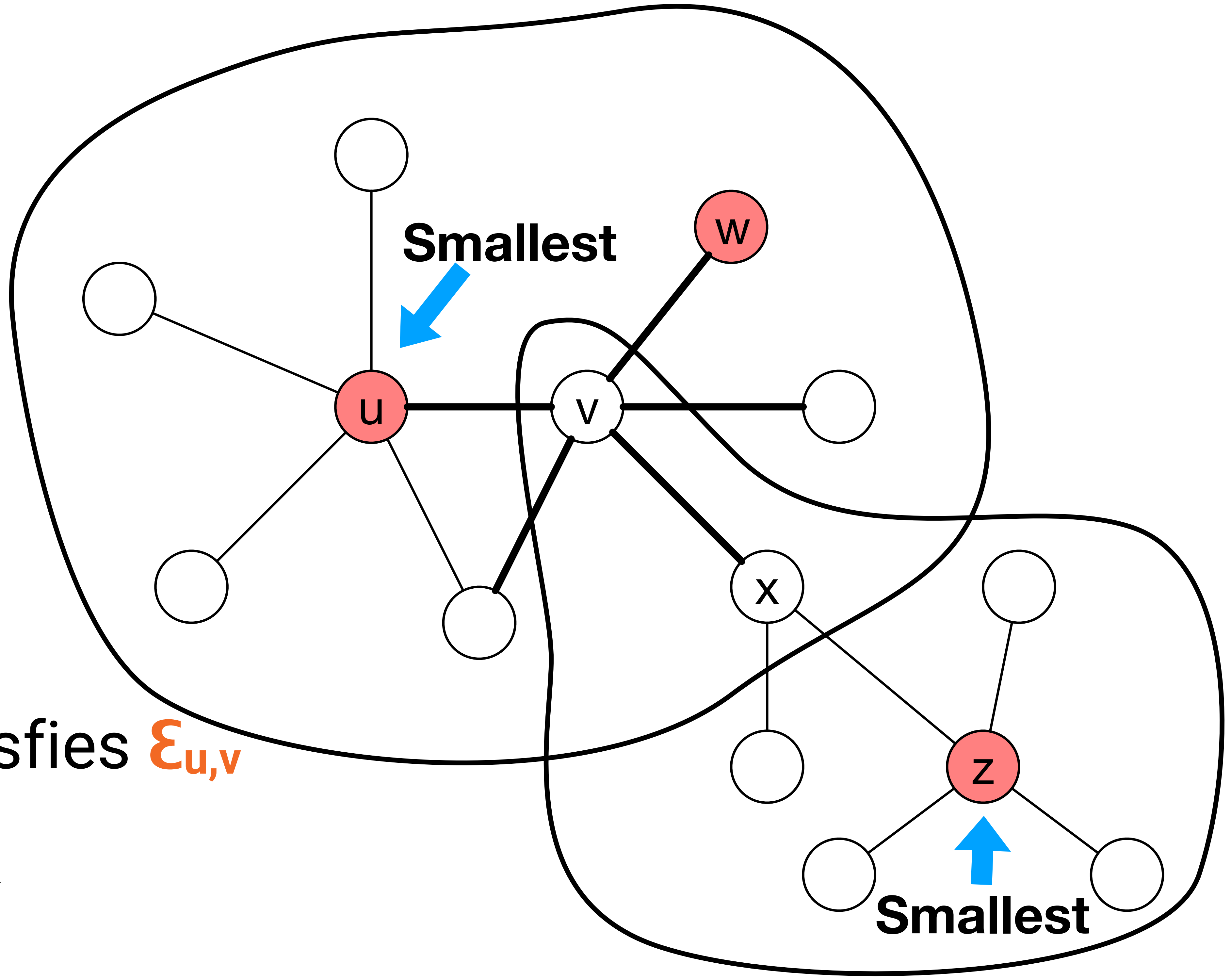
  - so every edge incident to v is counted once, **for v**
  - every edge can be **counted once**, **for each endpoint**



Randomized MIS: Luby's Algorithm

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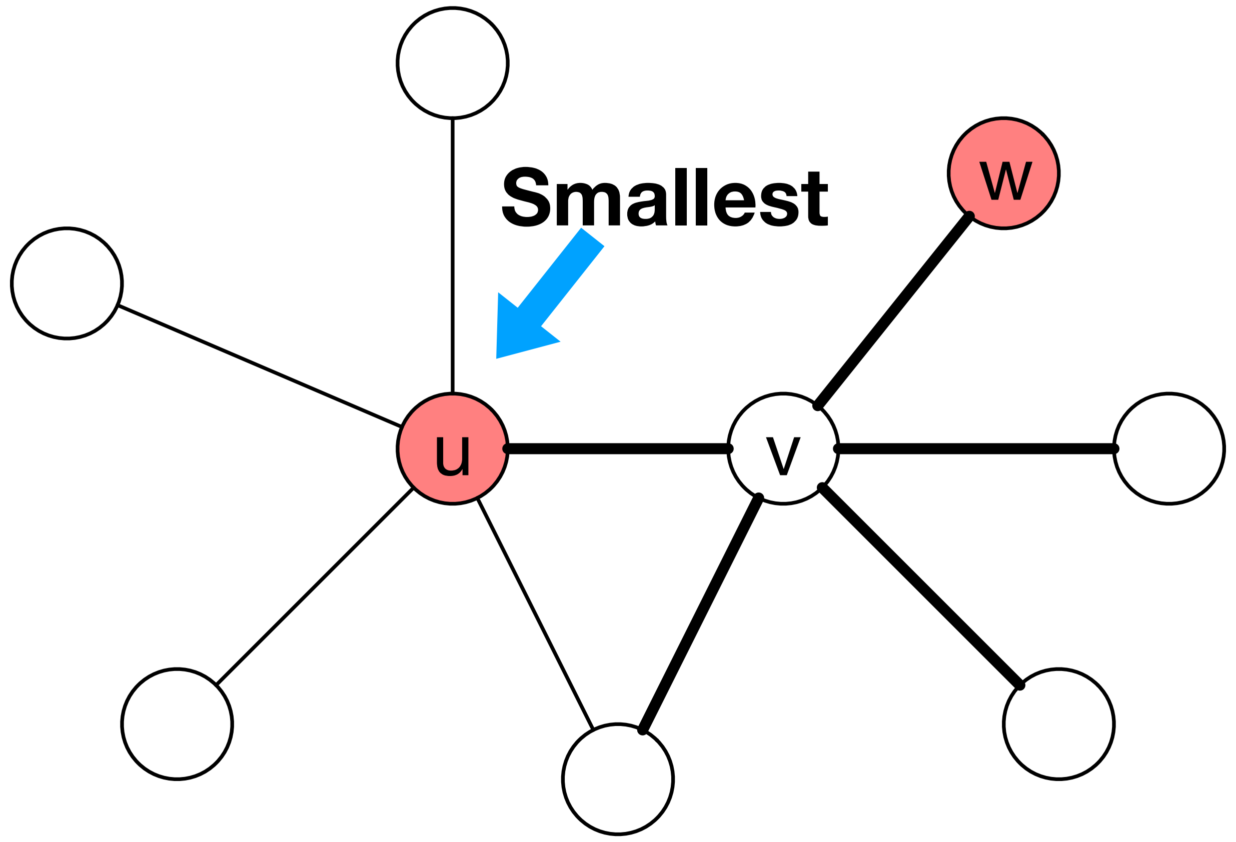

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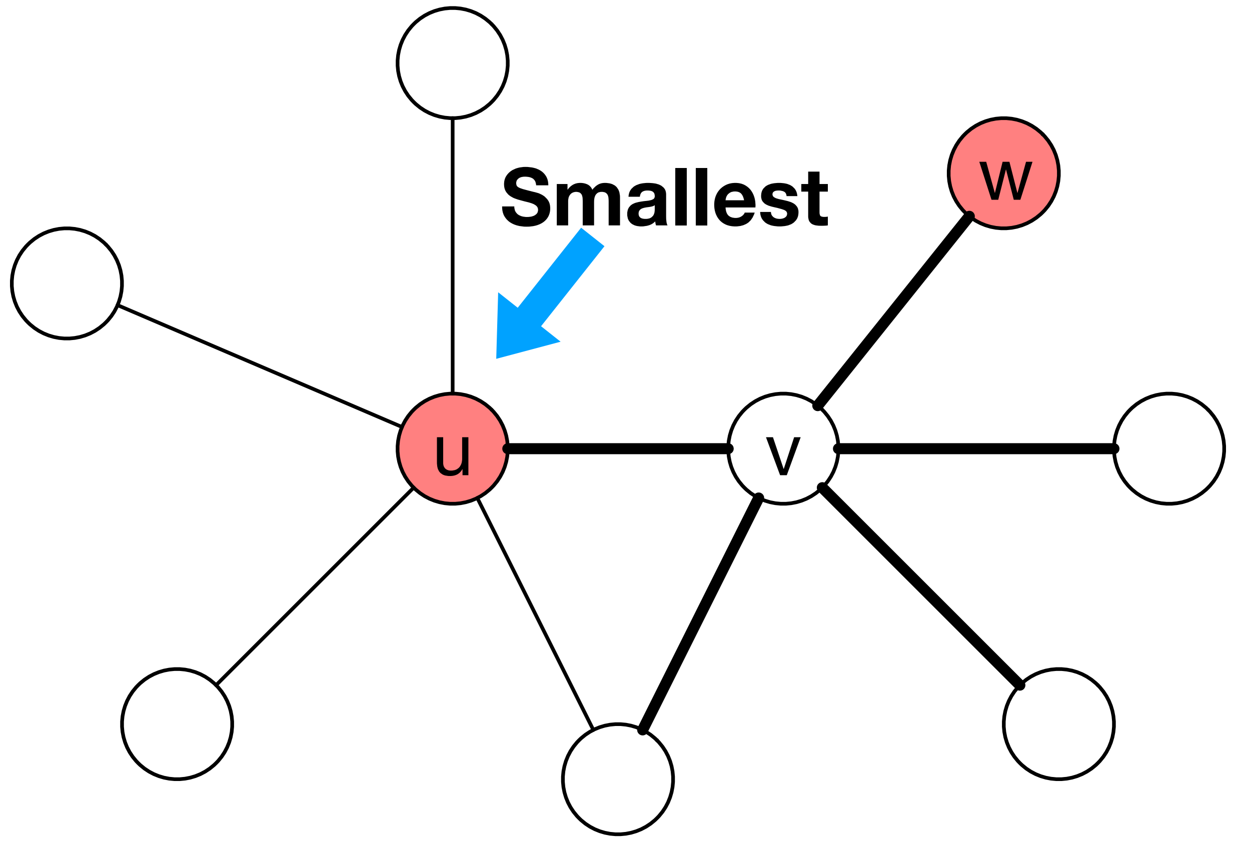

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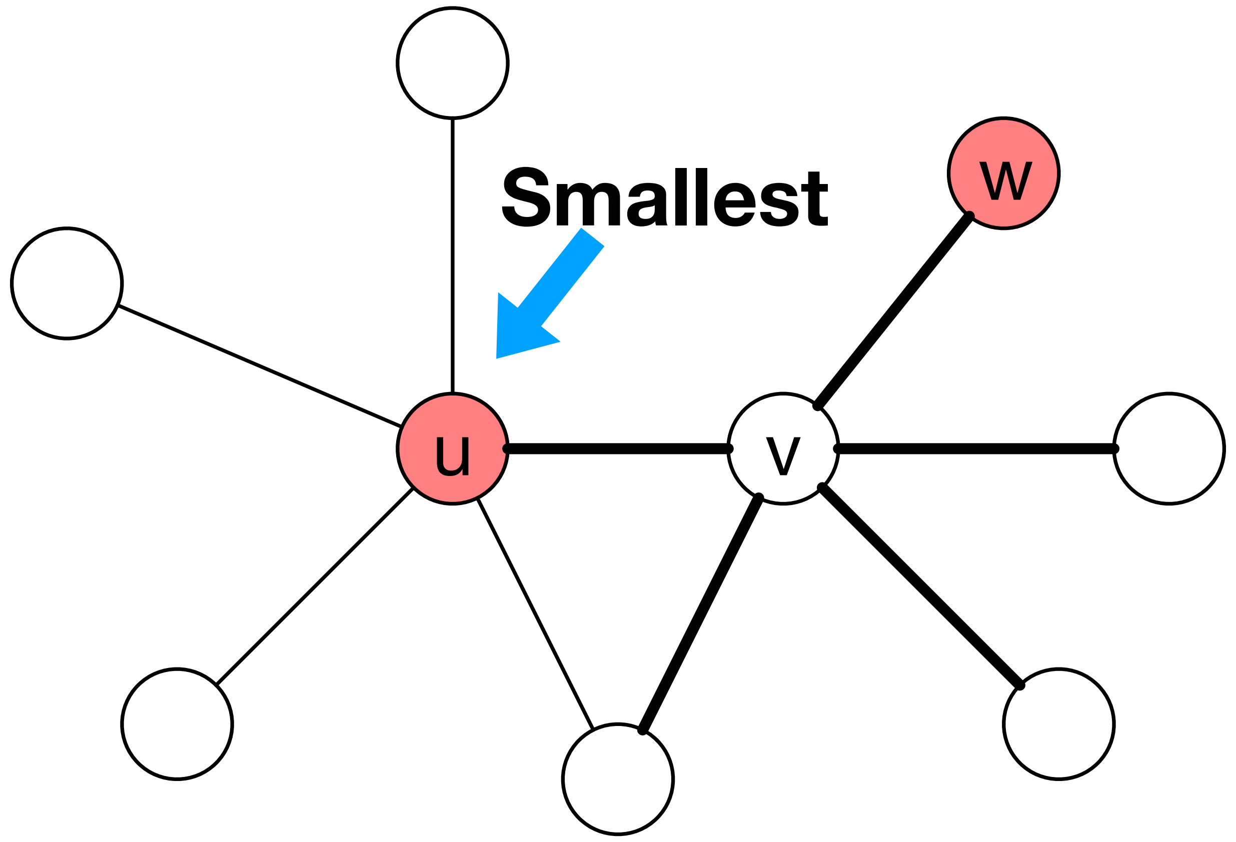

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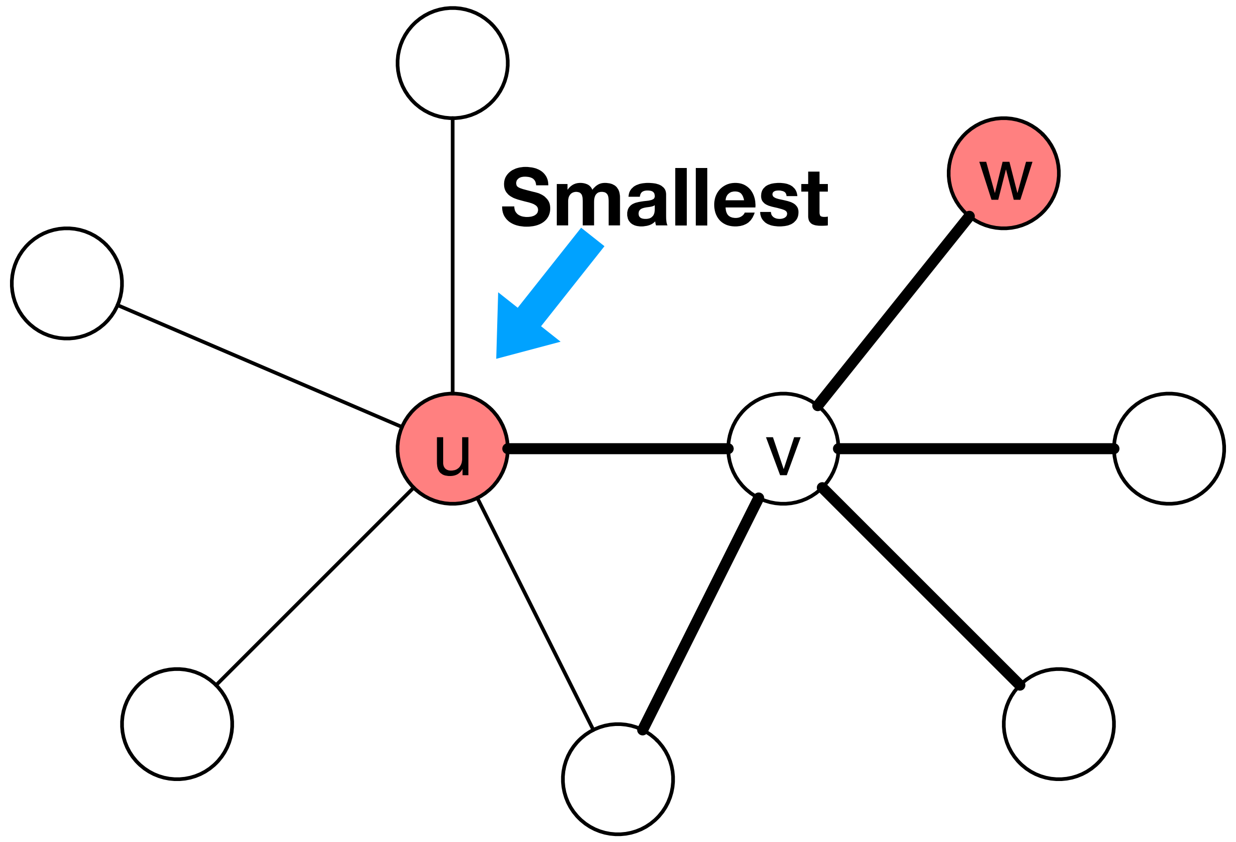

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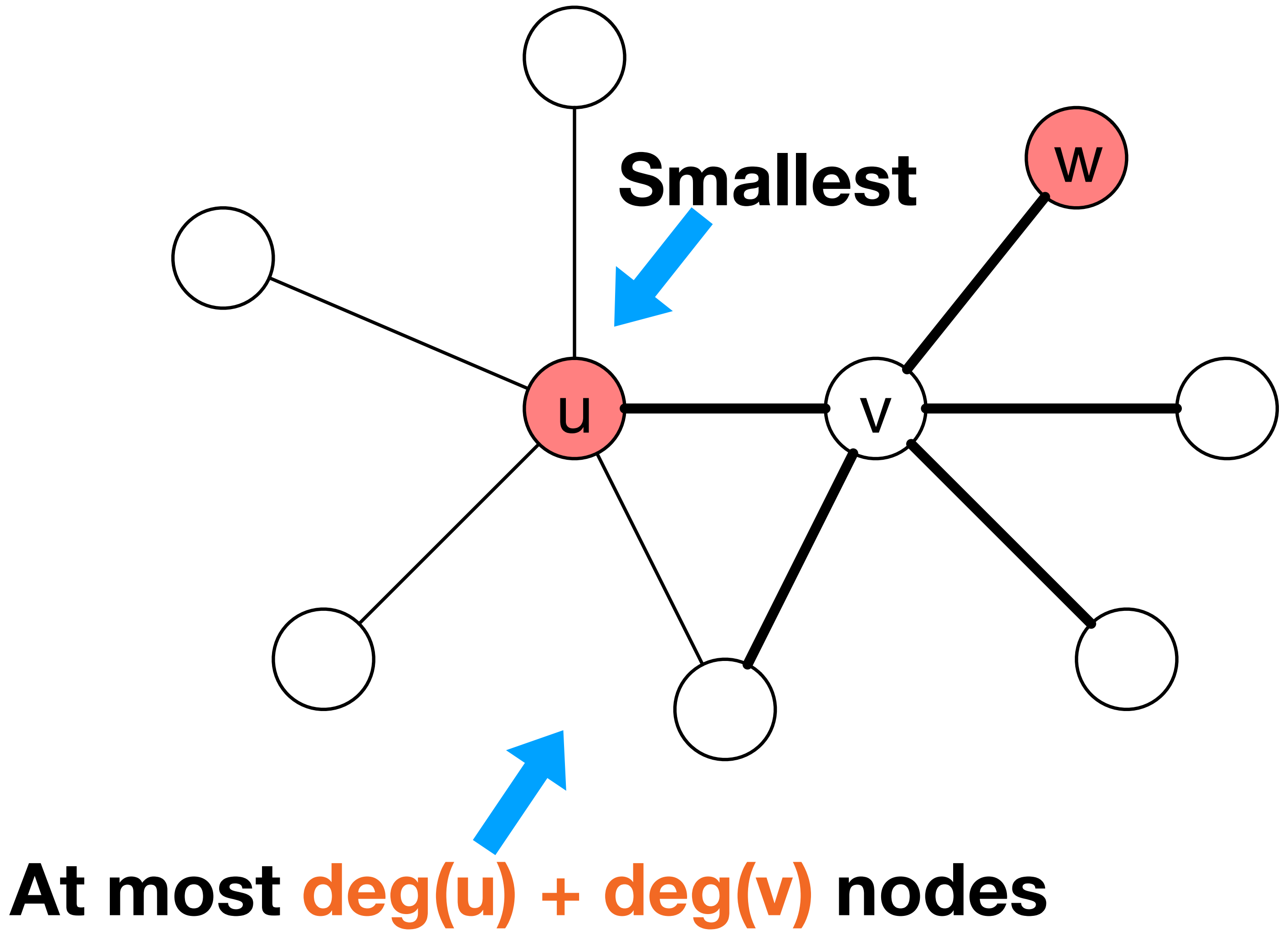

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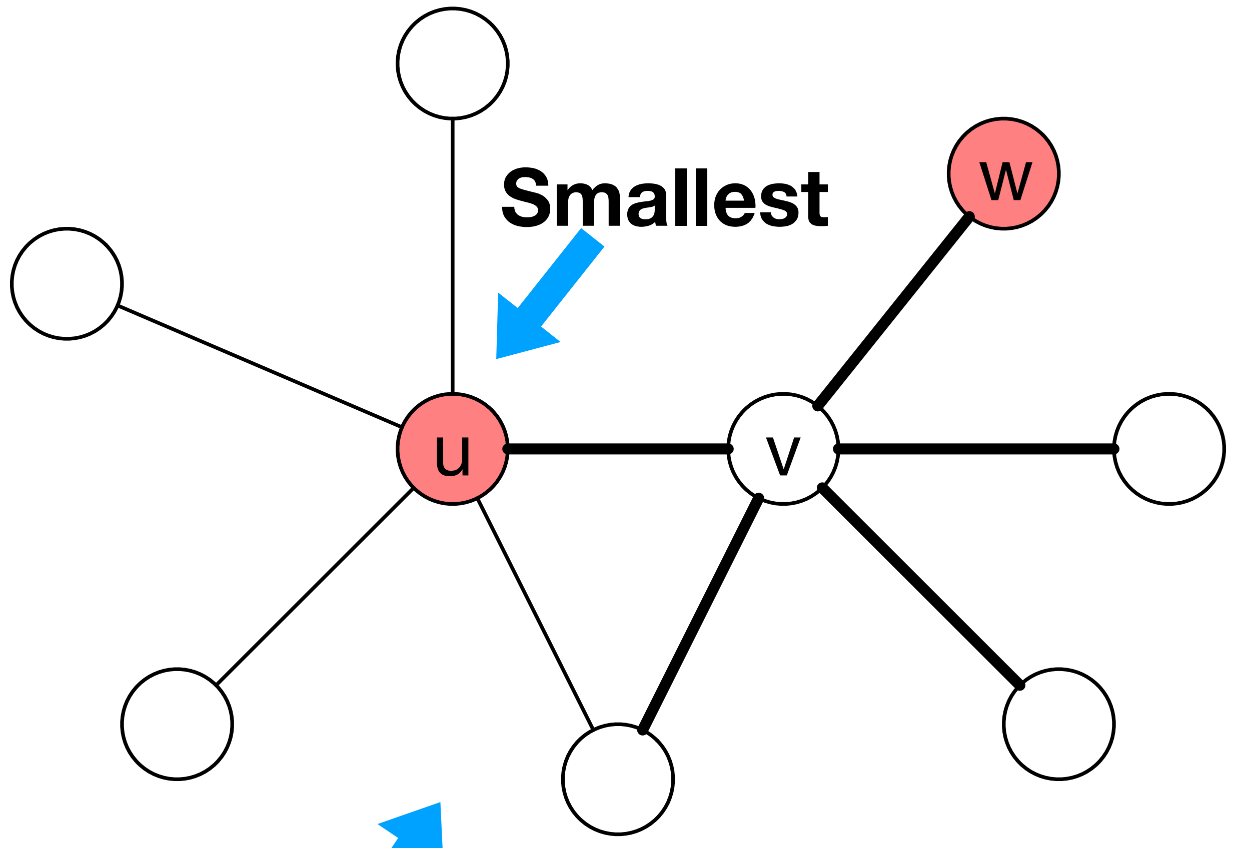

Smallest

At most deg(u) + deg(v) nodes

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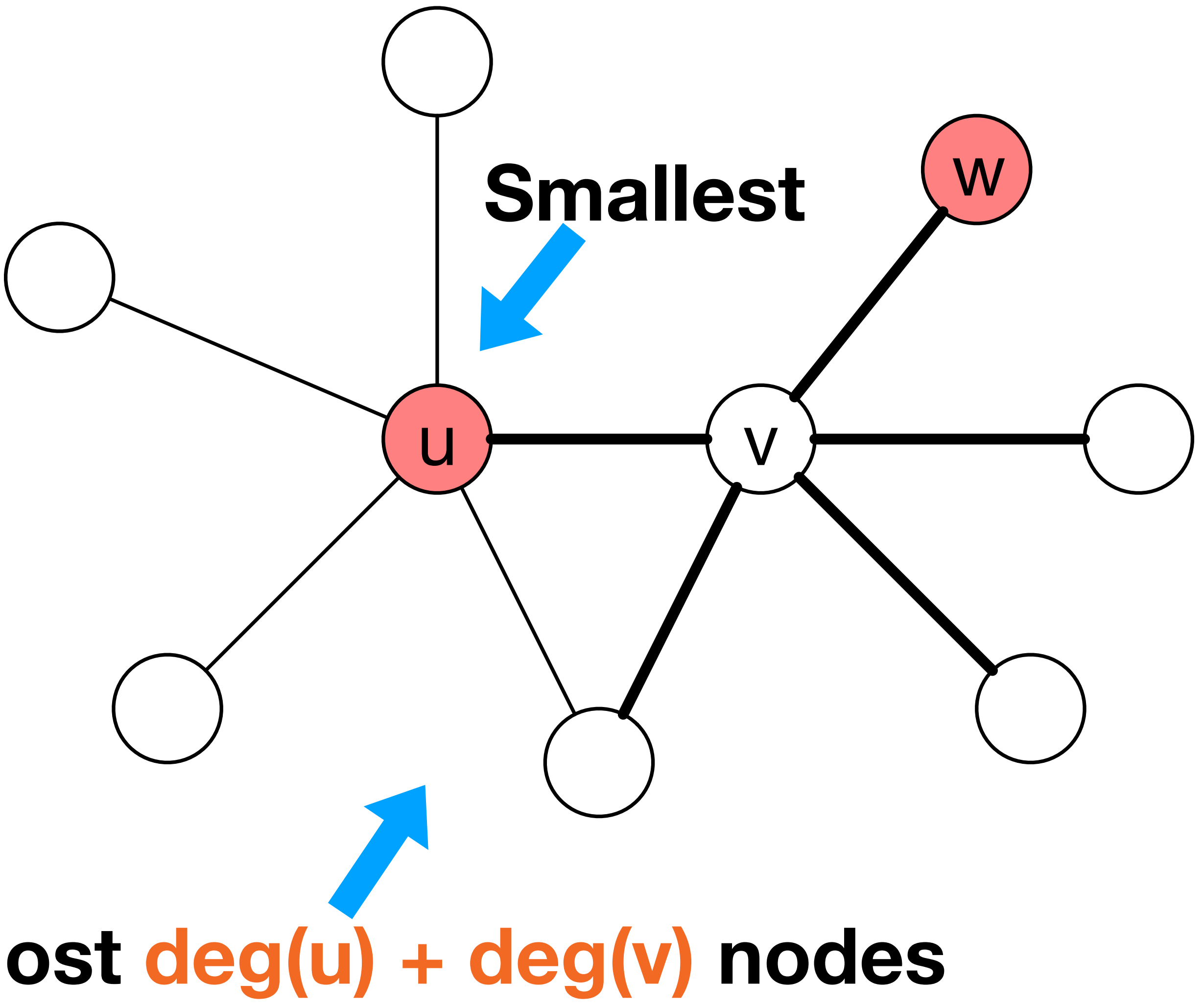

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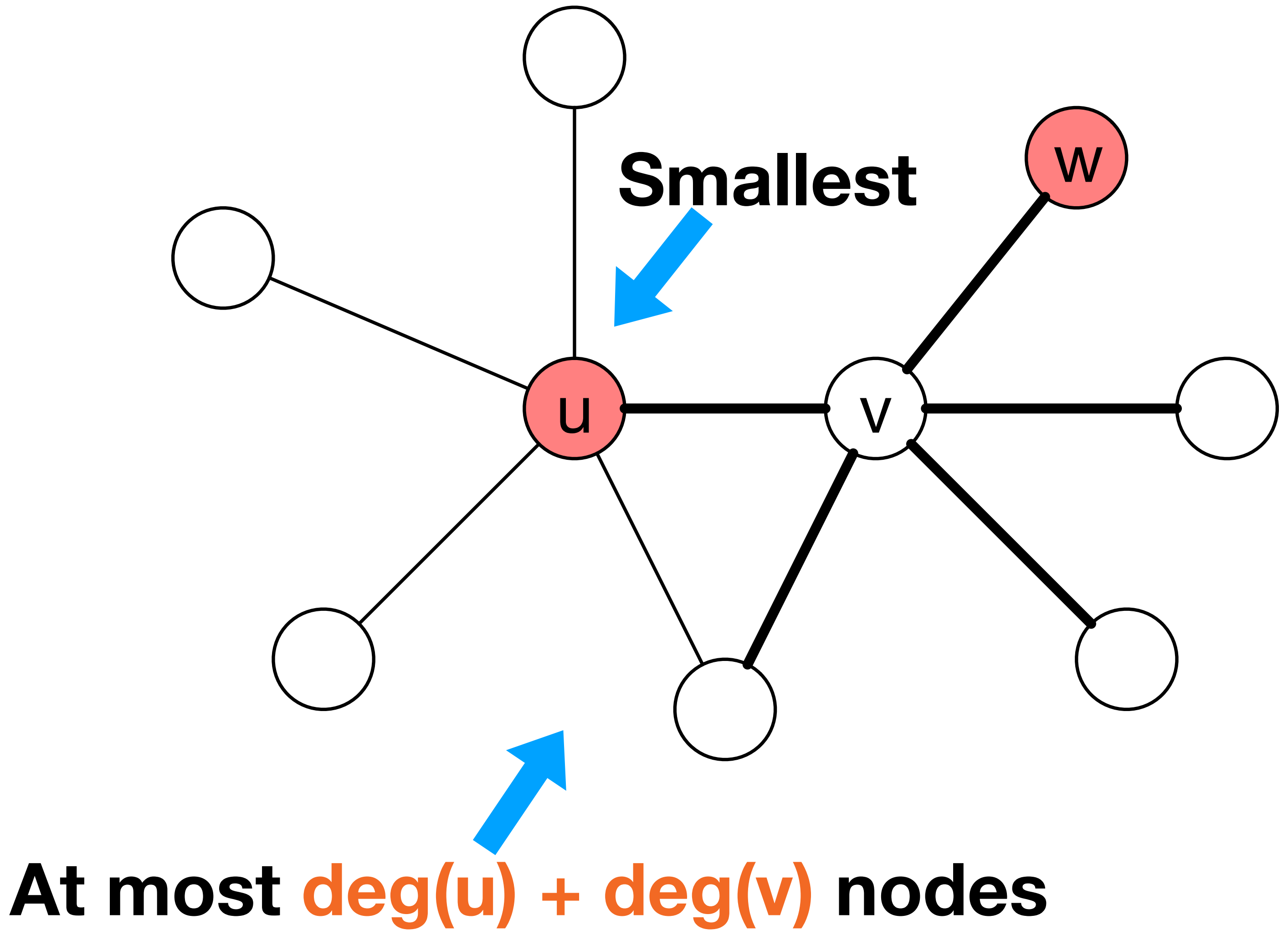

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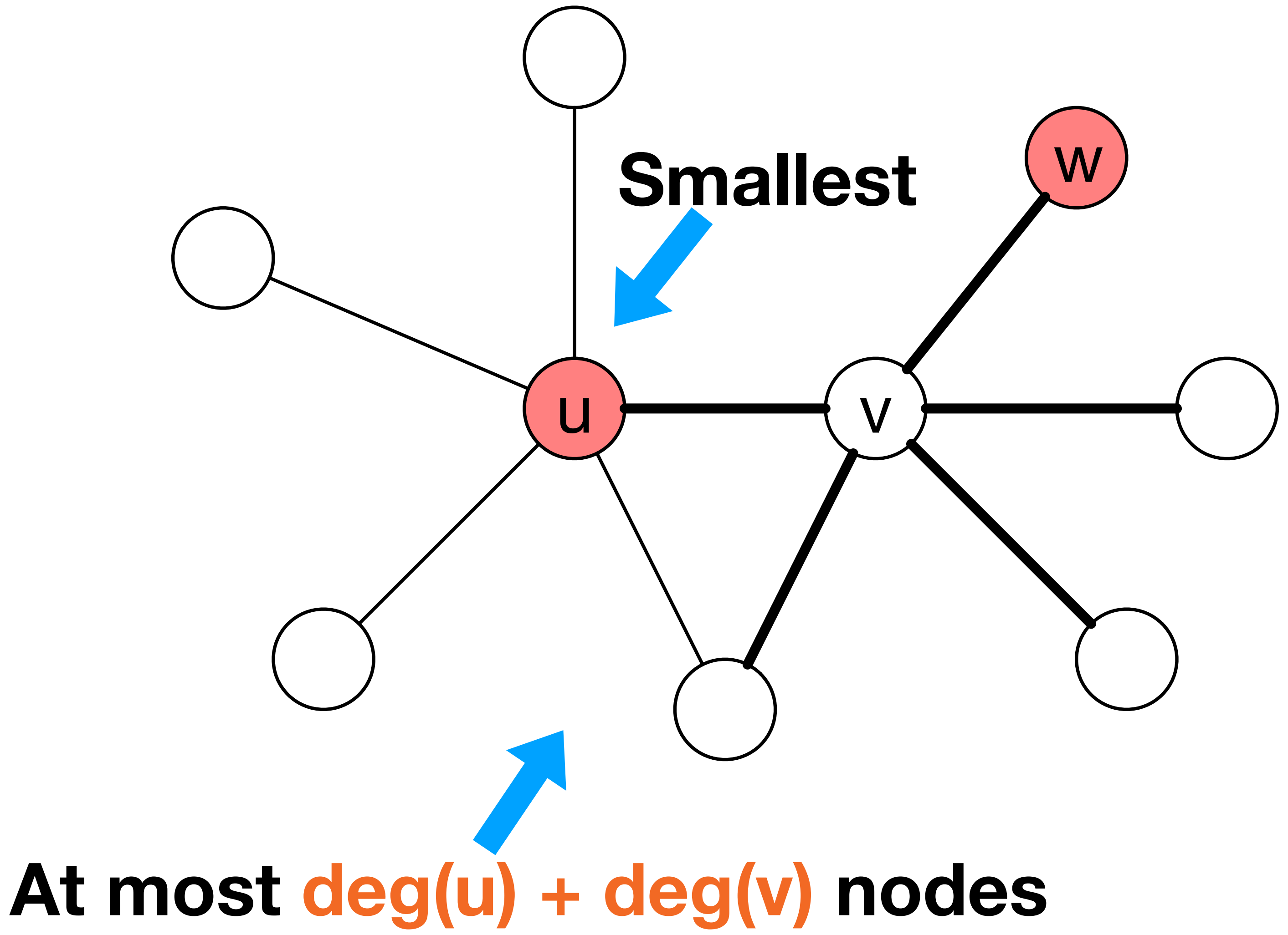

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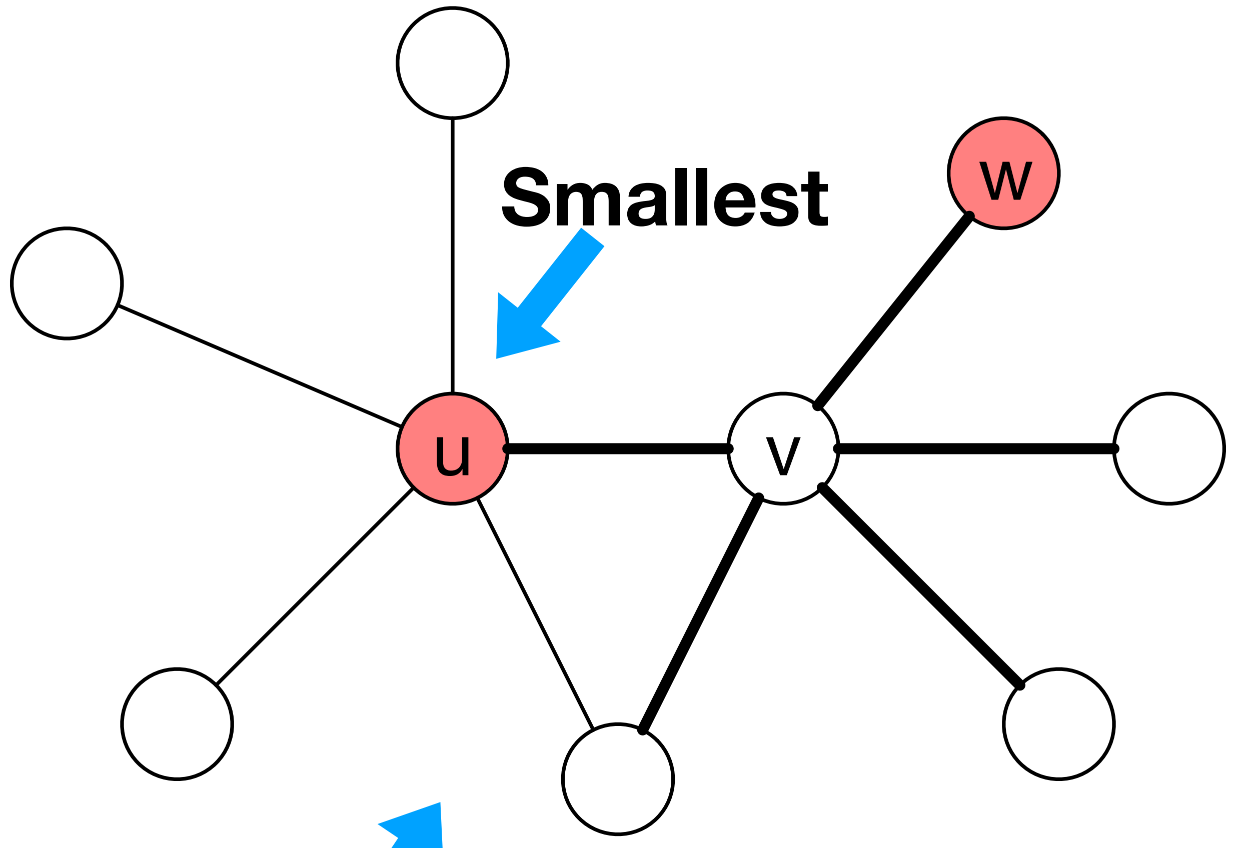

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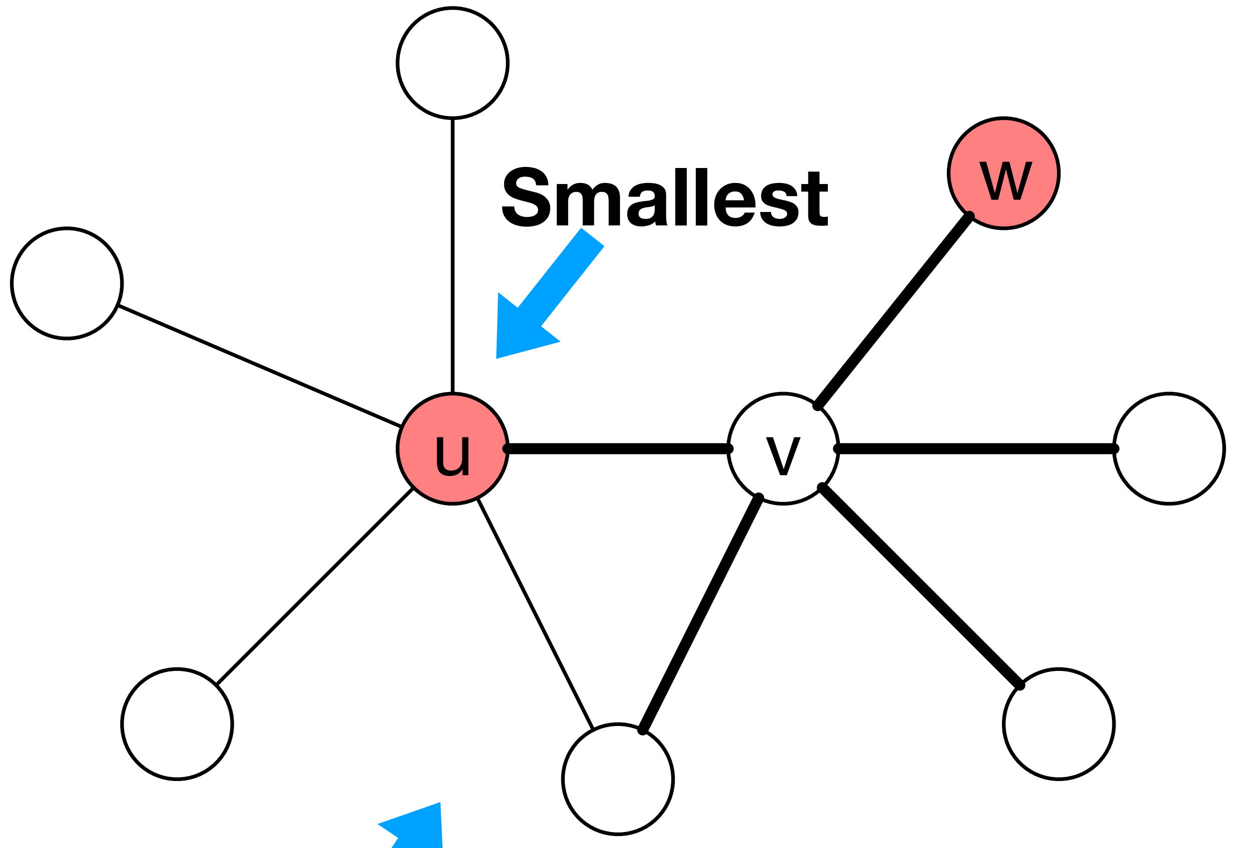

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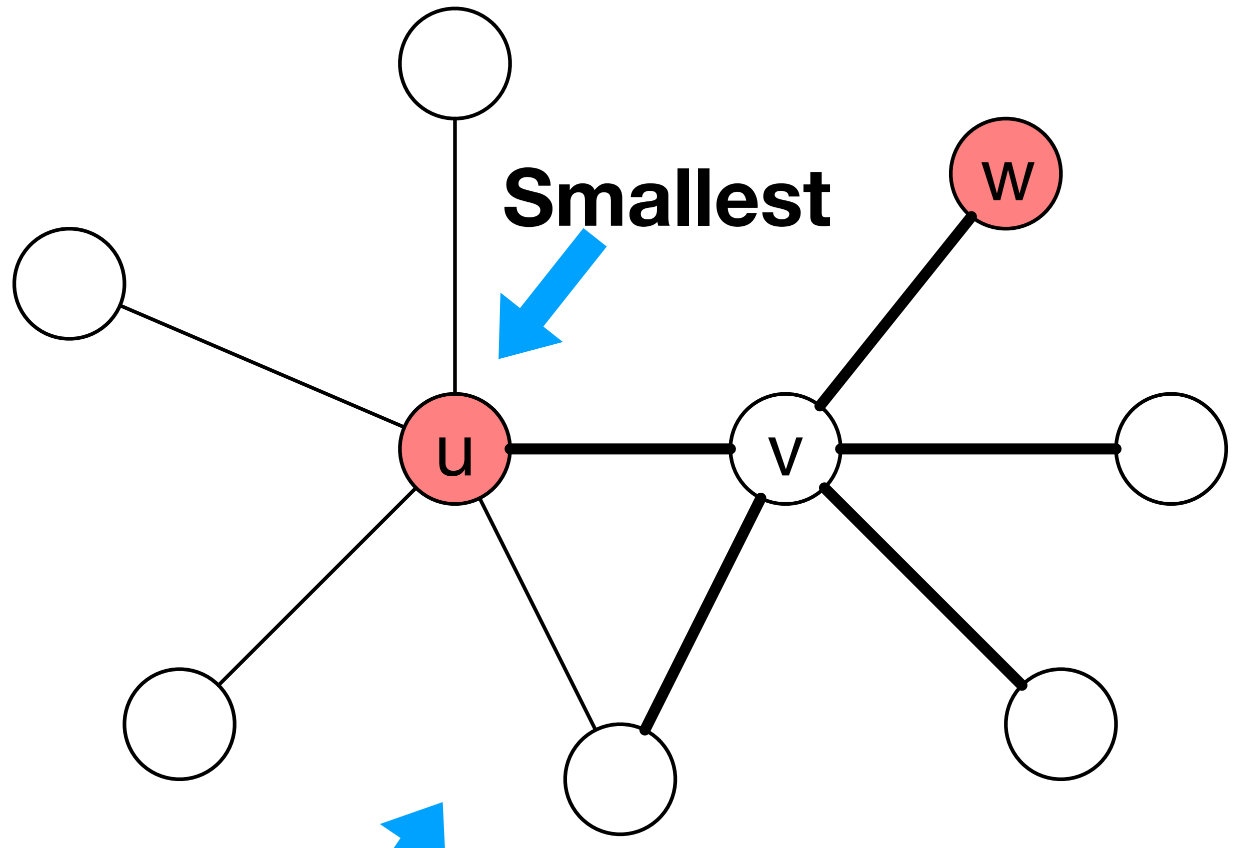

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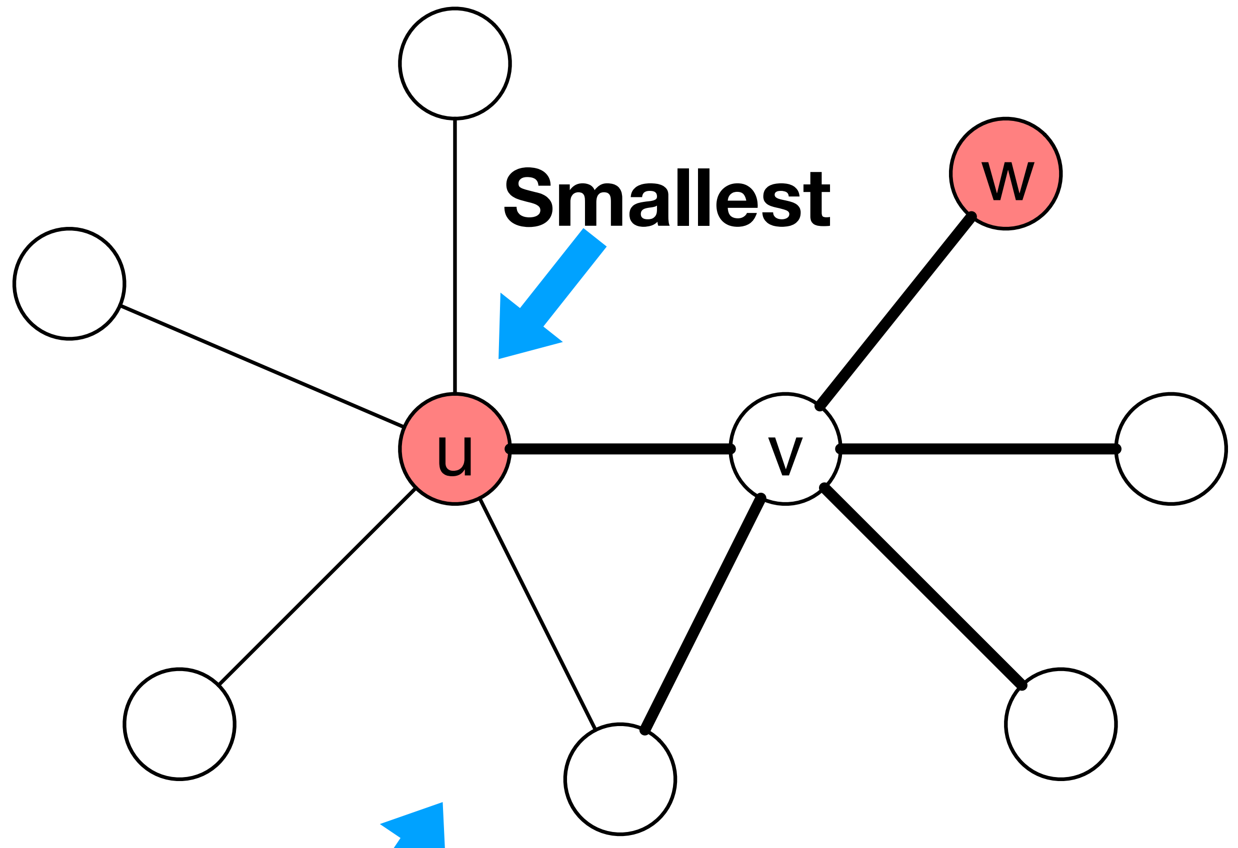

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From MIS to (Δ+1)-coloring

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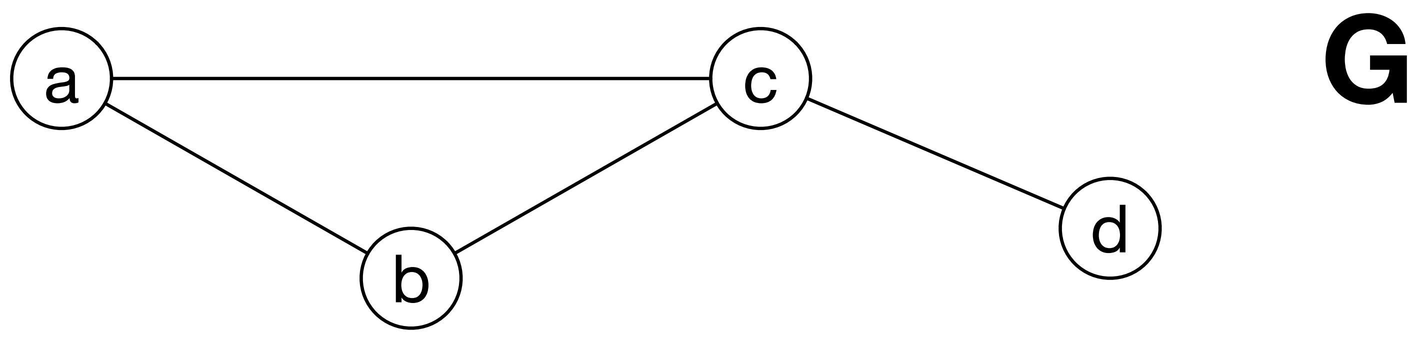

- Assume we want to compute a **coloring** on graph **G**

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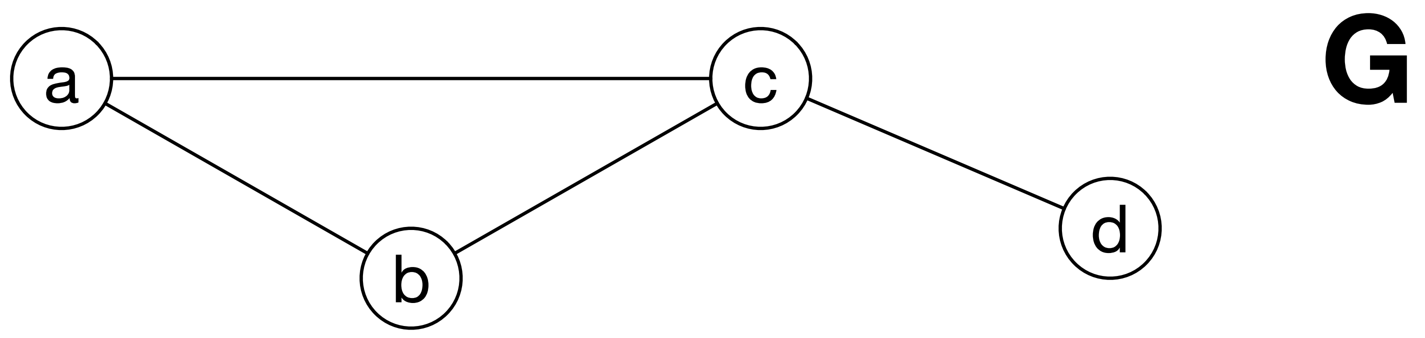

- Assume we want to compute a coloring on graph G
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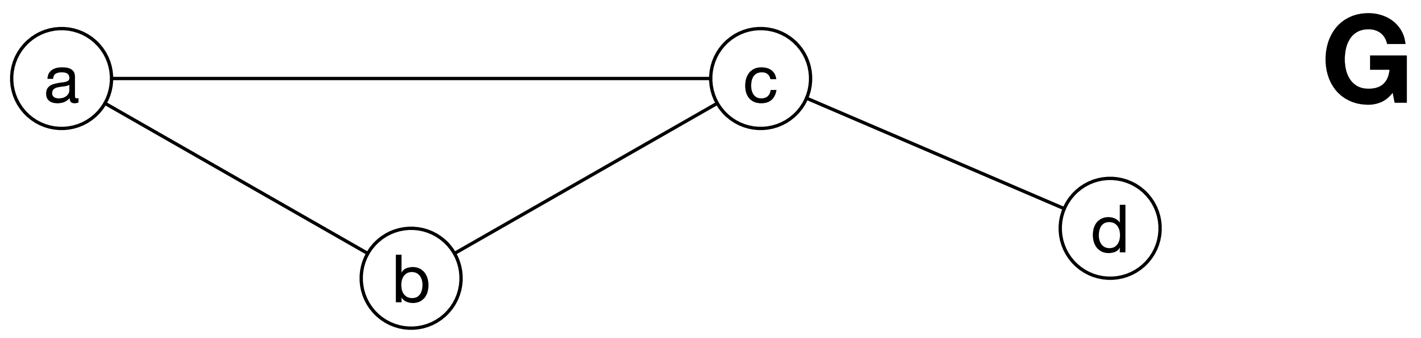
From MIS to (Δ+1)-coloring



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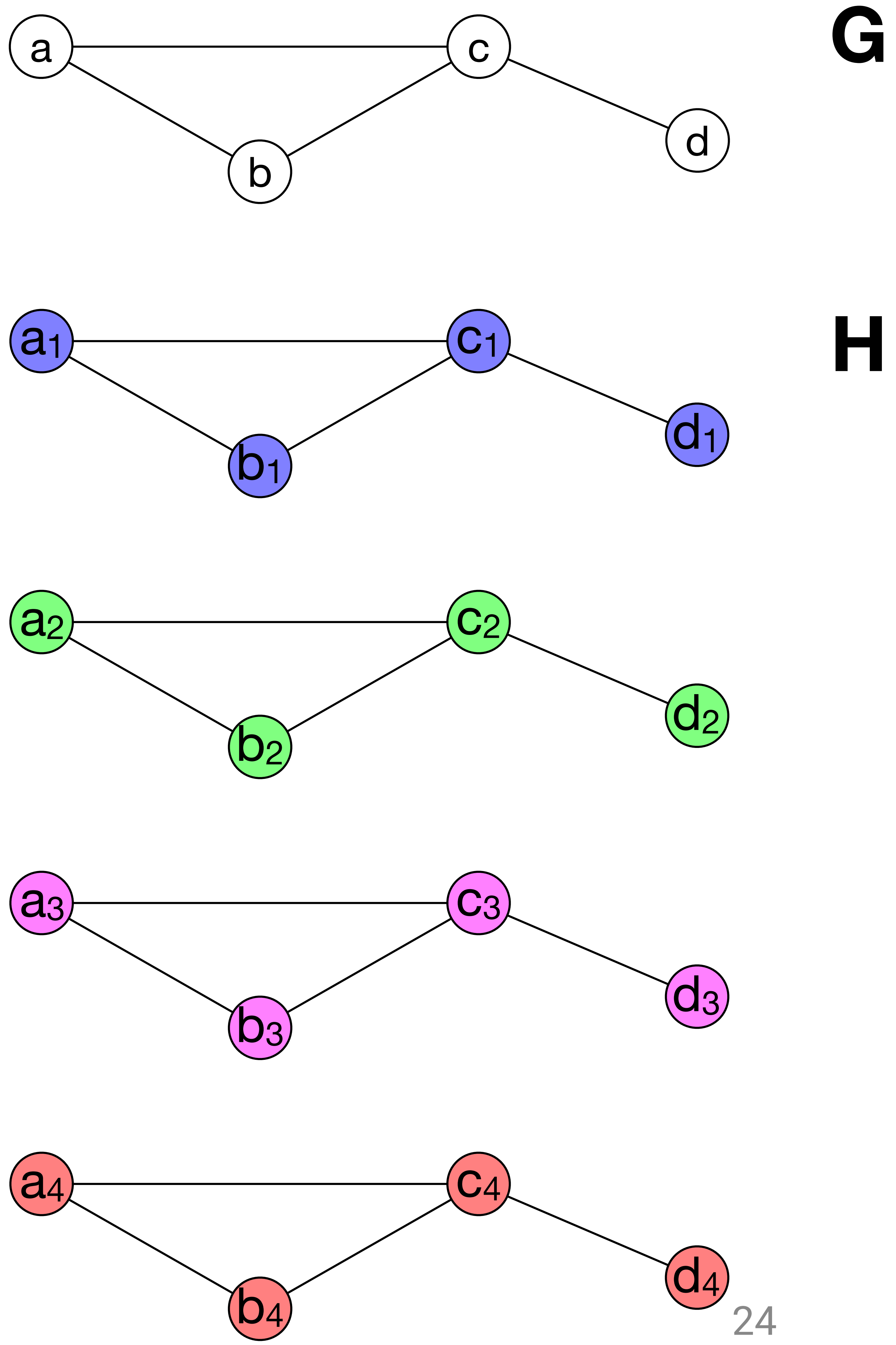

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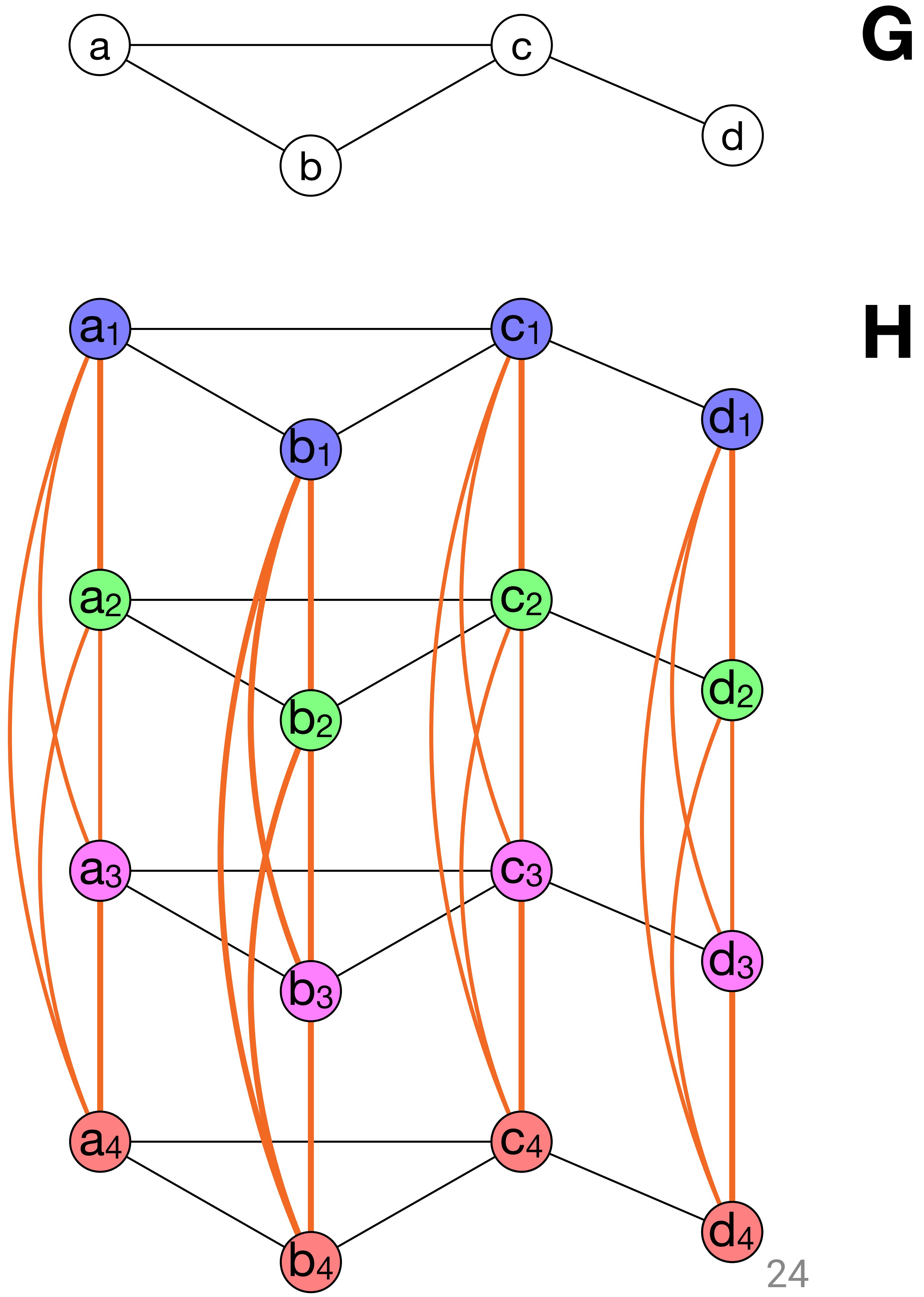

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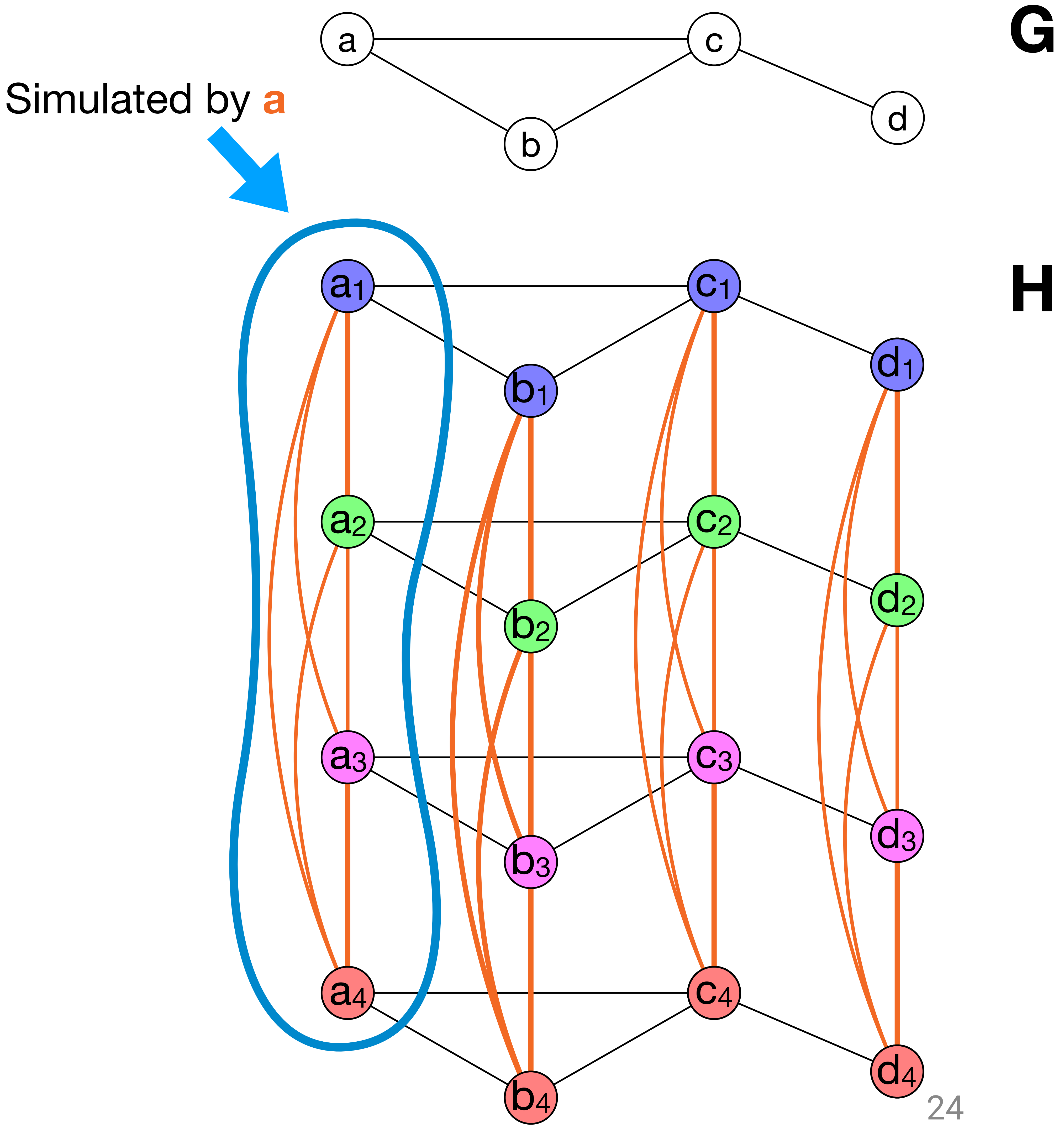

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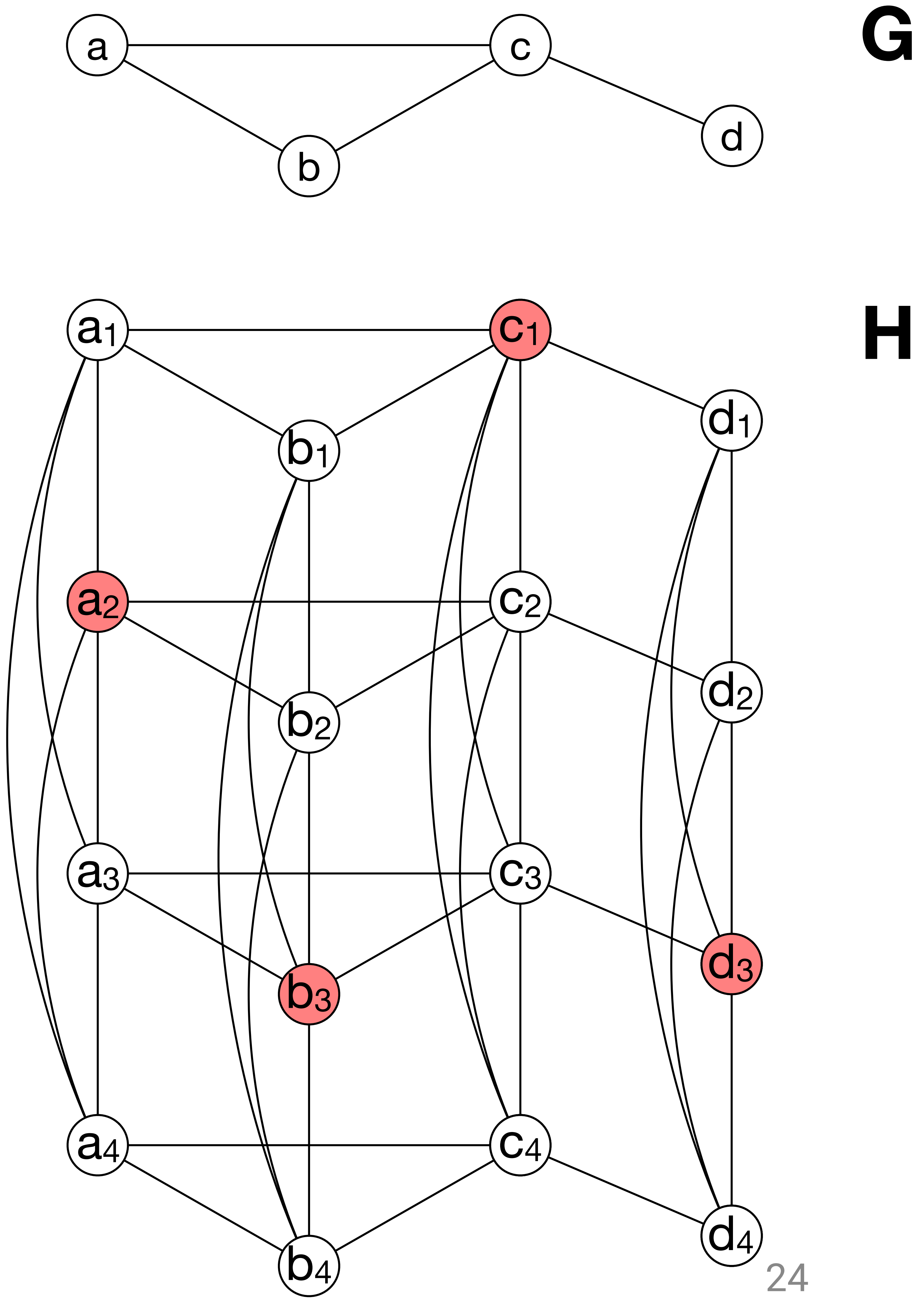
3. Compute **MIS** on **H**



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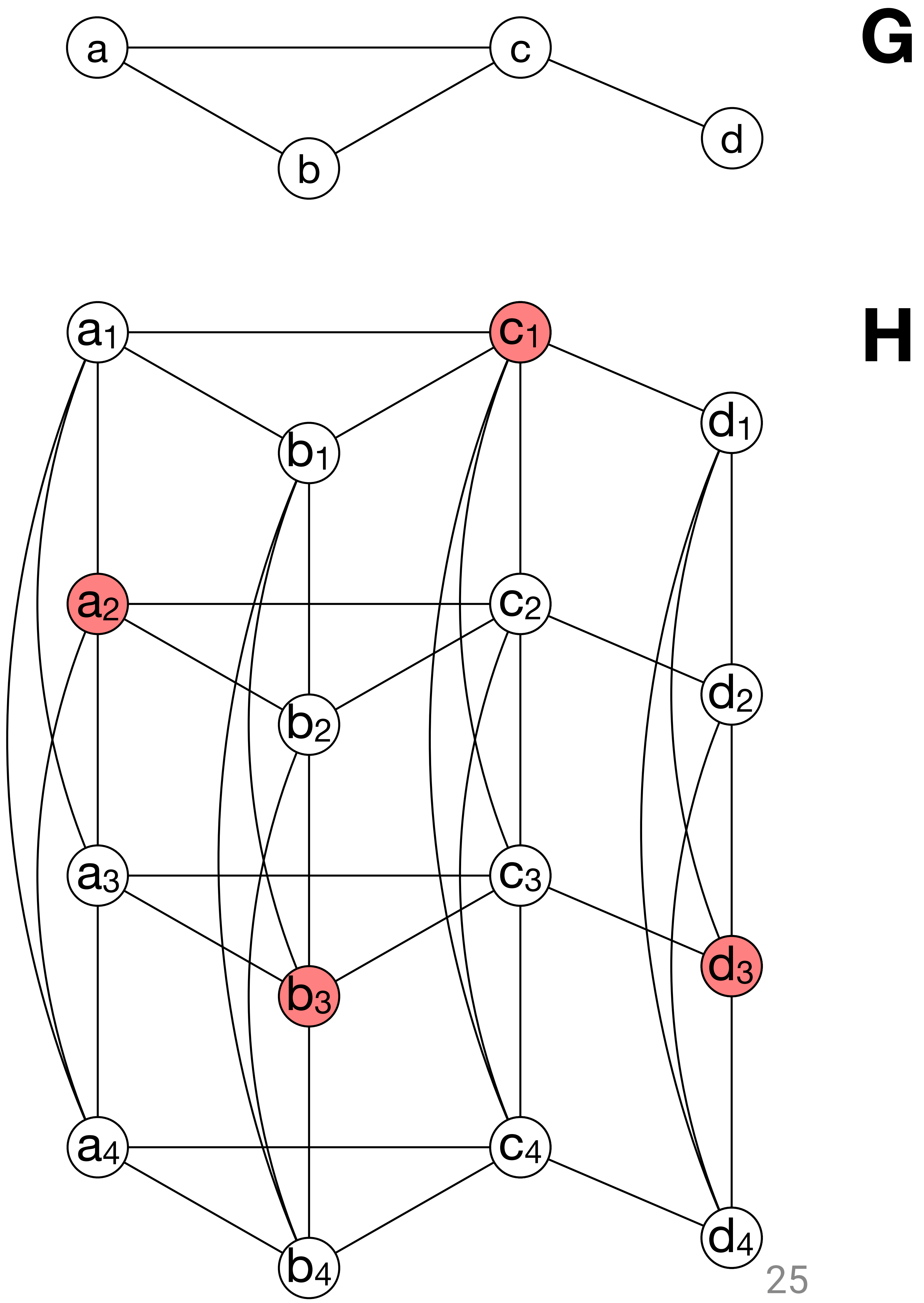
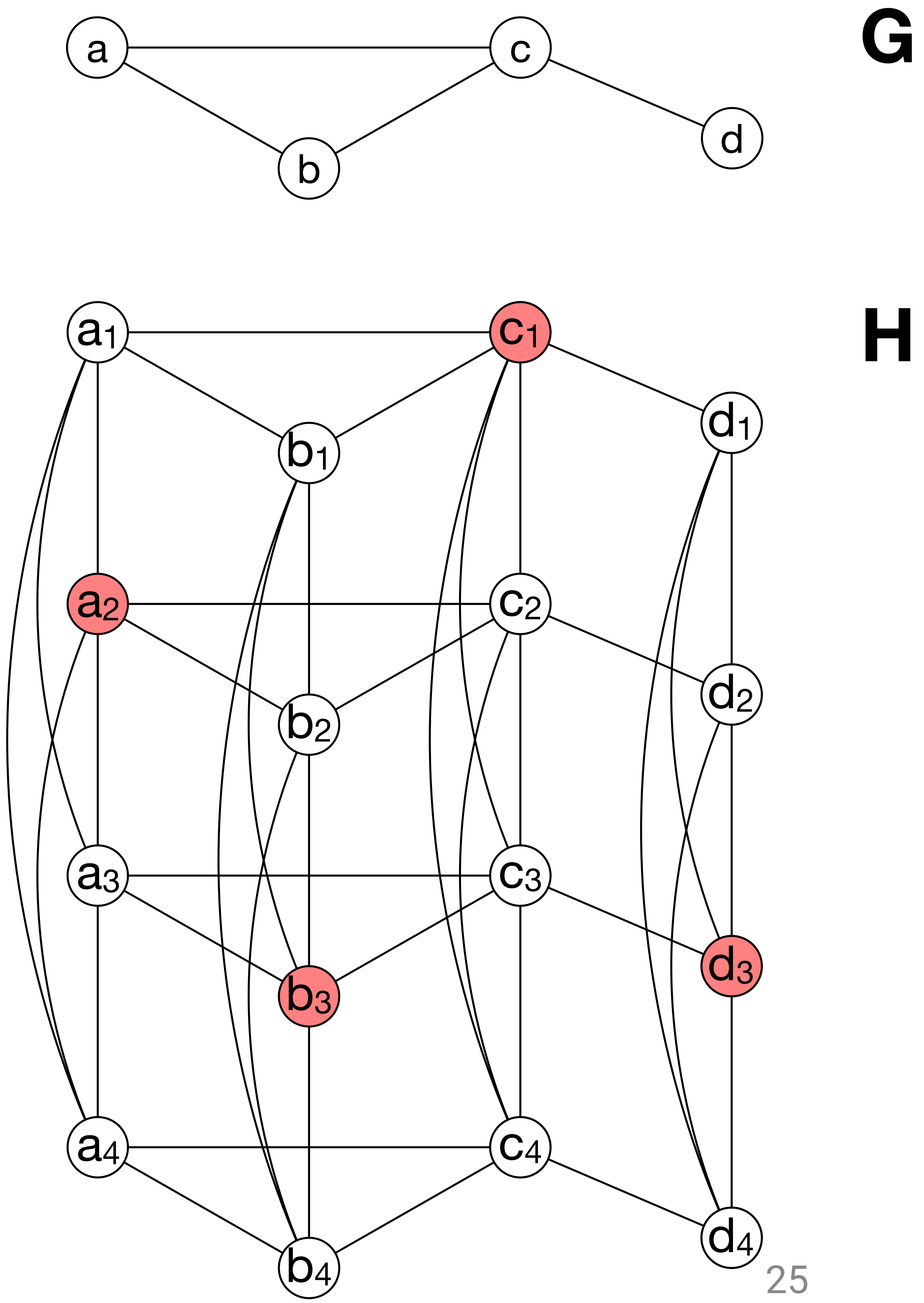


From MIS to (Δ+1)-coloring

- **Claim**: the MIS contains exactly one node from each column



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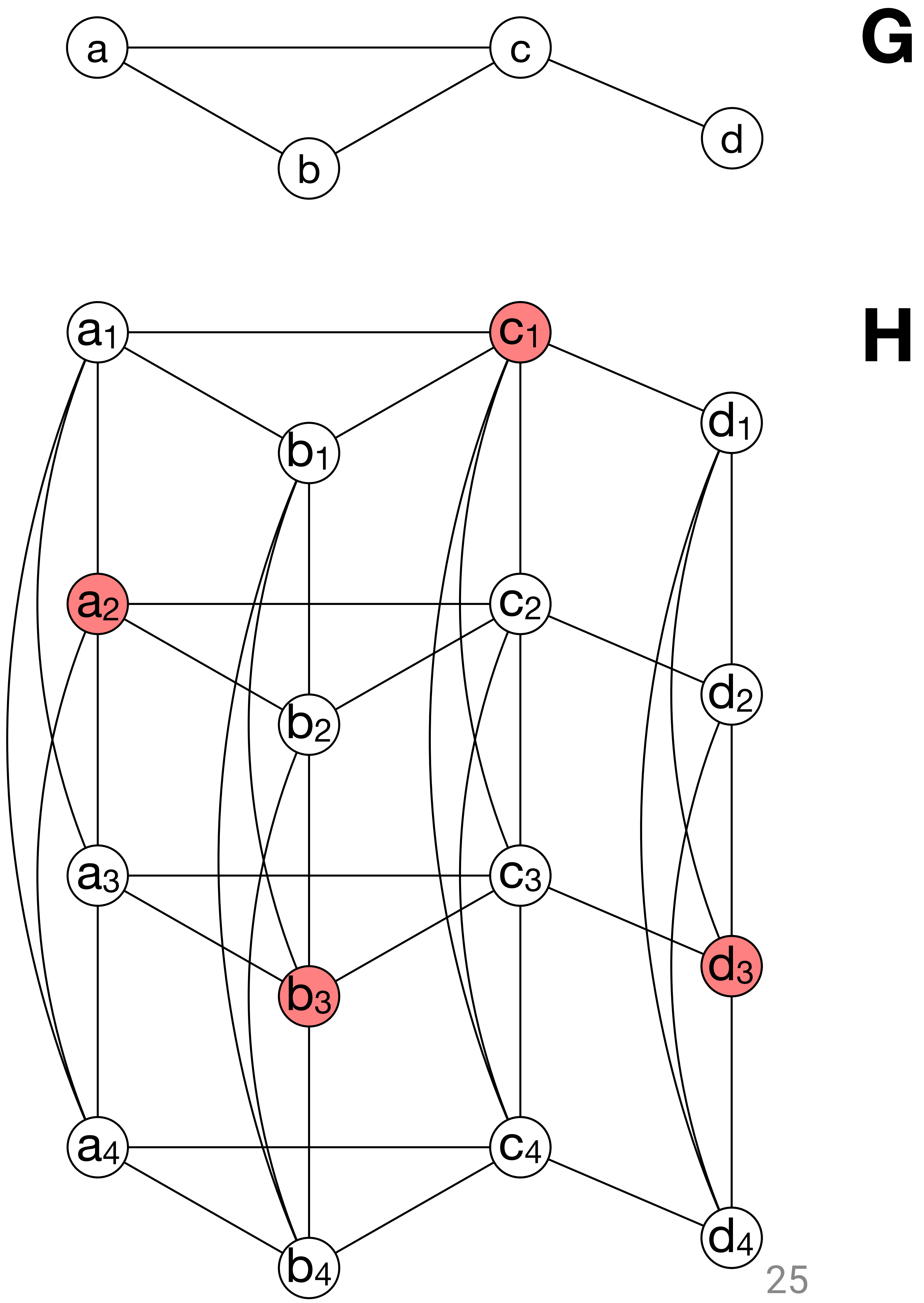

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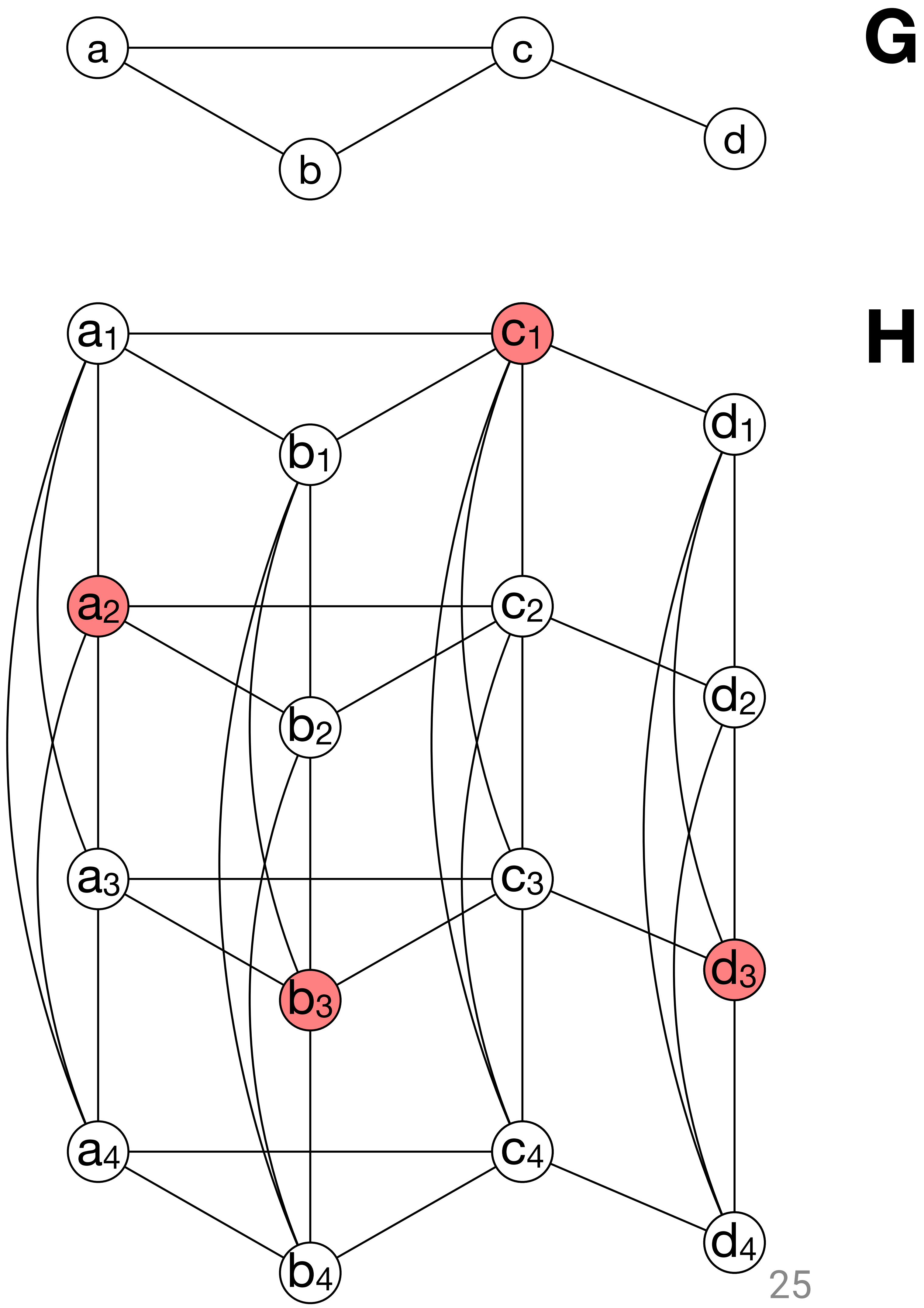

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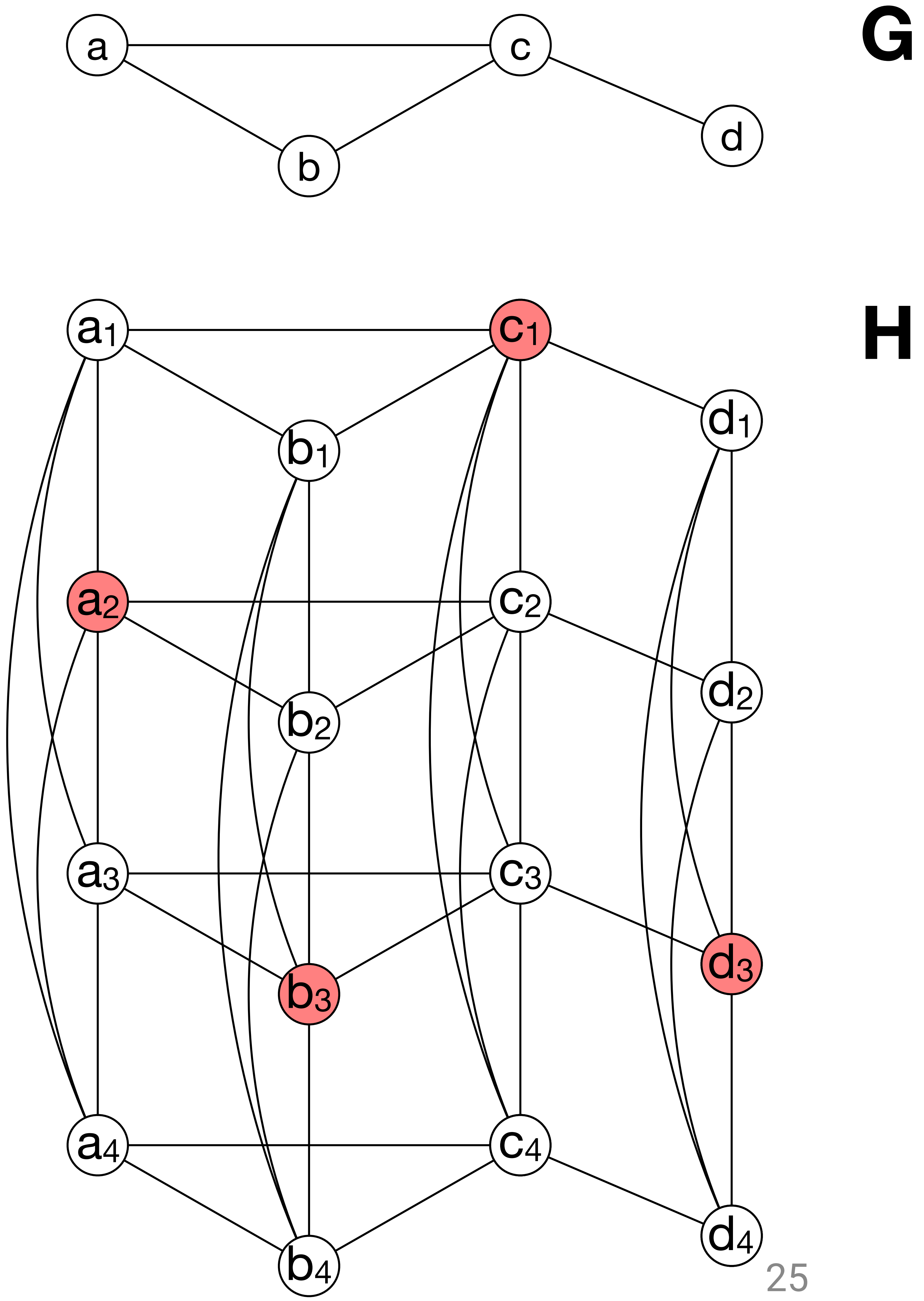
- **Algorithm**: if a **column node v_i** is in the **MIS**, node v picks **color i**



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  - Put **deg(v)+1** copies of **v** instead of **Δ+1**

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