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Theoretical Computer Science - Bridging Course
Exercise Sheet 8

Due: Monday, 18th of June 2024, 12:00 pm

Exercise 1: Class $\mathcal{P}$ *(1+3+3 Points)*

$\mathcal{P}$ is the set of languages ($\cong$ decision problems) which can be decided by an algorithm whose runtime can be bounded by $p(n)$, where p is a polynomial and n the size of the respective input (problem instance). Show that the following languages are in the class $\mathcal{P}$. Since it is typically easy (i.e. feasible in polynomial time) to decide whether an input is well-formed, your algorithm only needs to consider well-formed inputs. Use the $\mathcal{O}$-notation to bound the run-time of your algorithm.

(a) $\textsc{Palindrome} := \{w \in \{0,1\}^* \mid w \text{ is a Palindrome}\}$

(b) $4\text{-}\textsc{Clique} := \{\langle G\rangle \mid G \text{ has a } \textit{clique} \text{ of size at least } 4\}$

(c) $5\text{-}\textsc{VertexCover} := \{\langle G\rangle \mid G \text{ has a } \textit{vertex cover} \text{ of size at most } 5\}$.

Remarks:

- In both problems G is an undirected, simple graph.
- A *clique* in a graph $G = (V, E)$ is a set $C \subseteq V$ such that for all $u, v \in C : \{u, v\} \in E$.
- A *vertex cover* of $G = (V, E)$ is a subset $C \subseteq V$ of nodes, such that for all $\{u, v\} \in E$ it holds that $u \in C$ or $v \in C$.

Exercise 2: The Class $\mathcal{NP}$ *(Points)*

Show that the following problems (languages) are in class $\mathcal{NP}$.

(a) Given a graph $G = (V, E)$ and an integer k, it is required to determine whether G contains a clique of size at least k, hence consider the following problem:
$\textsc{Clique} := \{\langle G, k\rangle \mid G \text{ has a clique of size at least } k\}$.

(b) A *hitting set* $H \subseteq \mathcal{U}$ for a given universe $\mathcal{U}$ and a set $S = \{S_1, S_2, \ldots, S_m\}$ of subsets $S_i \subseteq \mathcal{U}$, fulfills the property $H \cap S_i \neq \emptyset$ for $1 \leq i \leq m$ (H 'hits' at least one element of every S_i).

Given a universe set $\mathcal{U}$, a set S of subsets of $\mathcal{U}$, and a positive integer k, it is required to determine whether $\mathcal{U}$ contains a hitting set of size at most k, hence consider the following problem:
$\textsc{HittingSet} := \{\langle \mathcal{U}, S, k\rangle \mid \text{universe } \mathcal{U} \text{ has subset of size} \leq k \text{ that } \textit{hits} \text{ all sets in } S \subseteq 2^{\mathcal{U}}\}$.[1]

[1] The power set $2^{\mathcal{U}}$ of some ground set $\mathcal{U}$ is the set of *all subsets* of $\mathcal{U}$. So $S \subseteq 2^{\mathcal{U}}$ is a collection of subsets of $\mathcal{U}$.