



# Theoretical Computer Science - Bridging Course

## Exercise Sheet 9

Due: Wednesday, 26th of June, 13:00 pm

### Exercise 1: Class $\mathcal{NP}$ part 1

Let  $L_1, L_2$  be languages (problems) over alphabets  $\Sigma_1, \Sigma_2$ . Then  $L_1 \leq_p L_2$  ( $L_1$  is polynomially reducible to  $L_2$ ), iff a function  $f : \Sigma_1^* \rightarrow \Sigma_2^*$  exists, that can be calculated in polynomial time and

$$\forall s \in \Sigma_1^* : s \in L_1 \iff f(s) \in L_2.$$

Language  $L$  is called  $\mathcal{NP}$ -hard, if *all* languages  $L' \in \mathcal{NP}$  are polynomially reducible to  $L$ , i.e.

$$L \text{ is } \mathcal{NP}\text{-hard} \iff \forall L' \in \mathcal{NP} : L' \leq_p L.$$

The reduction relation ' $\leq_p$ ' is transitive ( $L_1 \leq_p L_2$  and  $L_2 \leq_p L_3 \Rightarrow L_1 \leq_p L_3$ ). Therefore, in order to show that  $L$  is  $\mathcal{NP}$ -hard, it suffices to reduce a known  $\mathcal{NP}$ -hard problem  $\tilde{L}$  to  $L$ , i.e.  $\tilde{L} \leq_p L$ .

Finally a language is called  $\mathcal{NP}$ -complete ( $\Leftrightarrow L \in \mathcal{NPC}$ ), if

1.  $L \in \mathcal{NP}$  and
2.  $L$  is  $\mathcal{NP}$ -hard.

(a) Show  $\text{CLIQUE} := \{ \langle G, k \rangle \mid G \text{ has a clique of size at least } k \} \in \mathcal{NPC}$ .

(b) Show  $\text{HITTINGSET} := \{ \langle \mathcal{U}, S, k \rangle \mid \text{universe } \mathcal{U} \text{ has subset of size at most } k \text{ that hits all sets in } S \subseteq 2^{\mathcal{U}} \} \in \mathcal{NPC}$ .

For both parts, use the fact that  $\text{VERTEXCOVER} := \{ \langle G, k \rangle \mid \text{Graph } G \text{ has a vertex cover of size at most } k \} \in \mathcal{NPC}$ .

*Hint: For the poly. transformation ( $\leq_p$ ) you have to describe an algorithm (with poly. run-time!) that transforms:*

*for part (a), an instance  $\langle G, k \rangle$  of VERTEXCOVER into an instance  $\langle G', k' \rangle$  of CLIQUE s.t. a vertex cover of size  $\leq k$  in  $G$  becomes a clique of  $G'$  of size  $\geq k'$  vice versa(!)*

*for part (b), an instance  $\langle G, k \rangle$  of VERTEXCOVER into an instance  $\langle \mathcal{U}, S, k' \rangle$  of HITTINGSET, s.t. a vertex cover of size  $\leq k$  in  $G$  becomes a hitting set of  $\mathcal{U}$  of size  $\leq k'$  for  $S$  and vice versa(!).*

### Exercise 2: Class $\mathcal{NP}$ part 2

1. Given a set  $U$  of  $n$  elements ('universe') and a collection  $S \subseteq \mathcal{P}(U)$  of subsets of  $U$ , a selection  $C_1, \dots, C_k \in S$  of  $k$  sets is called a *set cover* of  $(U, S)$  of size  $k$  if  $C_1 \cup \dots \cup C_k = U$ .

Show that the problem

$$\text{SETCOVER} := \{ \langle U, S, k \rangle \mid U \text{ is a set, } S \subseteq \mathcal{P}(U) \text{ and there is a set cover of } (U, S) \text{ of size } k \}$$

is NP-complete.

You may use that

$$\text{DOMINATINGSET} = \{ \langle G, k \rangle \mid G \text{ has a dominating set with } k \text{ nodes} \}.$$

is NP-complete.

2. Show  $\text{DOMINATINGSET} := \{\langle G, k \rangle \mid \text{Graph } G \text{ has a dominating set of size at most } k\} \in \mathcal{NP}\mathcal{C}$ .

Use that  $\text{VERTEXCOVER} := \{\langle G, k \rangle \mid \text{Graph } G \text{ has a vertex cover of size at most } k\} \in \mathcal{NP}\mathcal{C}$ .

*Remark: A **vertex cover** is a subset of nodes of  $G$  such that every edge of  $G$  is incident to a node in the subset.*

*Hint: Transform a Graph  $G$  into a Graph  $G'$  such that a vertex cover of  $G$  will result in a dominating set  $G'$  and vice versa(!). Note that a dominating set is not necessarily a vertex cover ( $G = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\})$ ) has the dominating set  $\{v_1, v_4\}$  which is not a vertex cover). Also a vertex cover is not necessarily a dominating set (consider isolated nodes).*