University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
Z. Parsaeian

# Theoretical Computer Science - Bridging Course
# Sample Solution Exercise Sheet 4

**Due:** Tuesday, 14th of May 2024, 12:00 pm

## Exercise 1: CFGs and PDAs *(5+3 Points)*

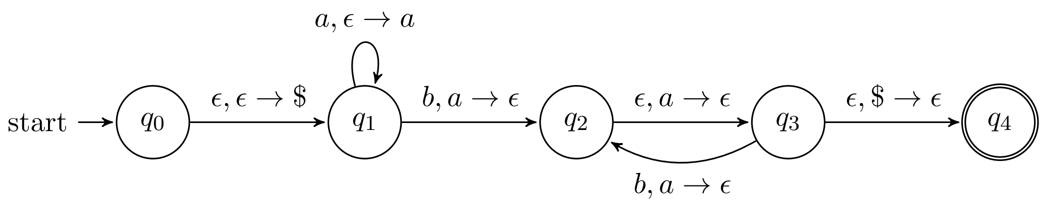
Give a context free grammar for each of the following languages.

1. 
   - $L_1 = \{a^i b^j | 0 < i \leq j\}$
   - $L_2 = \{a^{2n} b^n \mid n > 0\}$
   - $L_3 = \{a^* w c^k | w \in \{a, b\}^*,$ and $k$ is the number of $a$'s in $w\}$
2. Create a pushdown automaton that accepts languages $L_2$ and $L_3$.

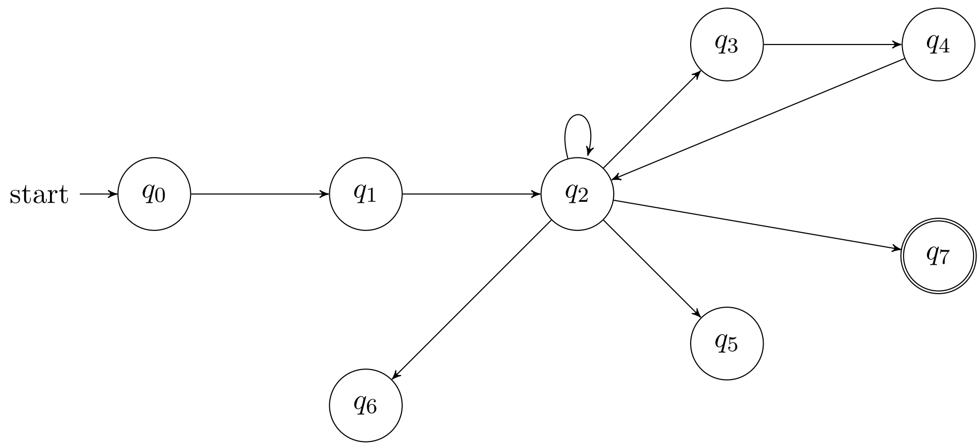
## Sample Solution

Let $S_1$, $S_2$, and $S_3$ be the respective start variables of the first three grammars.

1. 
   - $$\begin{aligned} S_1 &\to aBb \\ B &\to aBb \mid Bb \mid \epsilon \end{aligned}$$
   - $$S_2 \to aaS_2b \mid aab$$
   - $$\begin{aligned} S_3 &\to aS_3 \mid A \mid \epsilon \\ A &\to aAc \mid bA \mid \epsilon \end{aligned}$$
2. The formal definition of the automatons is implicitly given.
   PDA for $L_2$:



PDA for $L_3$:




## Exercise 2: Proving NonCFL *(4+4 Points)*

Use the Pumping Lemma to show that the following languages are not CFL.

1. $L_4 = \{a^n ba^{2n} ba^{3n} \mid n \geq 0\}$
2. $L_5 = \{a^i b^j c^k \mid i < j \text{ and } i < k\}$

*Bonus:* $L_6 = \{a^m \mid m \text{ is a prime}\}$
*NB: If you wish you can try first and prove it nonregular using the Pumping Lemma for regular languages and the same idea should be extended to CFLs.*

## Sample Solution

1. Assume the language is context free. This means that the property from the Pumping Lemma for context free languages should hold true for $L_4$.

   Now, let $p$ be any pumping length. Consider a string $s := a^p ba^{2p} ba^{3p}$ and let $s = uvxyz$ be a decomposition of $s$ with $|vxy| \leq p$ and $|vy| > 0$. We show that $uv^2xy^2z$ cannot be in the language, giving a contradiction. If $v$ or $y$ contained $b$, the string $uv^2xy^2z$ would have more than two $b$'s and is therefore not in the language. So assume that neither $v$ nor $y$ contains $b$. That means that $v$ as well as $y$ is fully contained in one of the three segments $a^p$, $a^{2p}$ and $a^{3p}$. But then pumping $s$ up to $uv^2xy^2z$ would violate the $1:2:3$ length ratio of the segments, because the length of at least one segment is changed (as $|vy| > 0$) and at least one segment keeps its length. Thus a contradiction to the Pumping Lemma. Therefore, $L$ is not context free.

2. Assume the language is context free. This means that the property from the Pumping Lemma for context free languages should hold true for $L_5$.

   Now, let $p$ be any pumping length. Consider a string $a^p b^{p+1} c^{p+1}$, which is in $L$ and has length greater than $p$. By the Pumping Lemma this must be representable as $uvxyz$, such that all $uv^ixy^iz$ are also in $L$. Neither $v$ nor $y$ may contain a mixture of symbols from $\{a, b, c\}$; otherwise they would be in the wrong order for $uv^2xy^2z$. Hence, suppose $v$ consists entirely of 'a's. Then there is no way $y$, which cannot have both 'b's and 'c's, can generate enough of these letters to keep their number greater than that of the 'a's (it can do it for one or the other of them, not both). Similarly $y$ cannot consist of just 'a's. So suppose then that $v$ or $y$ contains only 'b's or only 'c's. Consider the string $uv^0xy^0z$ which must be in $L$. Since we have dropped both $v$ and $y$, we must have at least one 'b' or one 'c' less than we had in $uvxyz$, which was $a^p b^{p+1} c^{p+1}$. Consequently, this string no longer has enough of either 'b's or 'c's to be a member of $L$. Thus a contradiction to the Pumping Lemma. Therefore, $L$ is not context free.

*Bonus solution:* Assume the language is context free. This means that the property from the Pumping Lemma for context free languages should hold true for $L_6$.
Now, let $p$ be any pumping length. Let $t > p$ be a prime. Since $a^t$ is in $L_2$, let $uvxyz$ be the decomposition of $s^t$, regarding the Pumping Lemma. Let $v = a^i$ and $y = a^j$ Note that $|vy| > 0$ and $|vxy| \leq p$. Hence, $i+j > 0$ It must hold that $uv^{t+1}xy^{t+1}z \in L_2$. However, $uv^{ti+1}xy^{tj+1}z = uvxyz \cdot v^t y^t = a^{t(1+i+j)}$. Since both $t$ and $1+i+j$ are greater than 1, $t(1+i+j)$ is not a prime. Hence, $uv^{ti+1}xy^{tj+1}z$ is not in $L_2$. Thus a contradiction to the Pumping Lemma. Therefore, $L$ is not context free.

*Remark:* $L_1$ can also be proven non regular using the pumping lemma for regular languages. Assume the language is regular This means that the property from the Pumping Lemma for regular languages should hold true for $L_6$.Let $p$ be the pumping length. Let $t > p$ be a prime. Then, let $x = 0^i$, $y = 0^j$ and $z = 0^k$ such that $x + y + z = t$. Based on Pumping Lemma, for all $\ell \geq 0$, $xy^\ell z$ must also be in $L_2$. However, for $\ell = t+1$, $xy^\ell z = 0^i 0^{j(t+1)} 0^k = 0^{(i+j+k)} 0^{tj} = 0^{t(j+1)}$, where both $t$ and $j+1$ are greater than 1. Therefore, $t(j+1)$ is not a prime, and hence $xy^{t+1}z$ is not in $L_2$. Thus a contradiction to the Pumping Lemma. Therefore, $L$ is not regular.

## Exercise 3: Closure in CFL *(2+2 Points)*

1. Show that the context-free languages are closed under union, concatenation and Kleene star. *Hint: try to prove that the context-free languages are closed under the above operators via creating the appropriate grammars.*

2. Knowing that $L_7 = \{a^i b^j c^k \mid i < j\}$ is a context free language, are context free languages closed under intersection?
*Hint: Use the fact that $L_5$ is not a context free language.*

## Sample Solution

1. The context-free languages are closed under union, concatenation and Kleene star, i.e. if $L_1$ and $L_2$ are context-free languages, so are $L_1 \cup L_2$, $L_1L_2$ and $L_1^*$. Indeed, we will prove that the languages are closed by creating the appropriate grammars. Suppose we have two context-free languages $L_1$ and $L_2$ , represented by grammars with start symbols $S_1$ and $S_2$ respectively. First of all, rename all the terminal symbols in the second grammar so that they don't conflict with those in the first. Then:

   To get the union, add the rule $S \rightarrow S_1 \mid S_2$, with $S$ representing the start symbol of the grammar of $L_1 \cup L_2$.
   To get the concatenation, add the rule $S \rightarrow S_1\ S_2$, with $S$ representing the start symbol of the grammar of $L_1L_2$.
   To get the Kleene star of $L_1$, add the rule $S \rightarrow S_1\ S \mid \varepsilon$ to the grammar for $L_1$, with $S$ representing the start symbol of the grammar of $L_1^*$.

2. The context-free languages are not closed under intersection, i.e. if $L_1$ and $L_2$ are context-free languages, it it not always true that $L_1 \cap L_2$ is also. We will prove the non-closure of intersection by exhibiting a counter-example. Consider the following two context free languages:

$$\begin{array}{rcl} L_7 & = & \{a^i b^j c^k \mid i < j\} \\ L_8 & = & \{a^i b^j c^k \mid i < k\} \end{array}$$

   The intersection of these languages is $L_7 \cap L_8 = \{a^i b^j c^k \mid i < j \textit{ and } i < k\} = L_5$ and we just proved that this language is not context-free.