



Theoretical Computer Science - Bridging Course

Sample Solution Exercise Sheet 6

Due: Tuesday, 4th of June 2024, 12:00 pm

Exercise 1: Decidable Problems

(2+2 +2+2 Points)

Show that the following languages are decidable.

1. $A = \{\langle R, S \rangle \mid R \text{ and } S \text{ are regular expressions and } L(R) \subseteq L(S)\}$.
2. Let $B = \{\langle G \rangle \mid G \text{ is a CFG over } \{0, 1\} \text{ and } 1^* \cap L(G) \neq \phi\}$. Use the fact that a language $C \cap R$ is context free for some context free language C and regular language R .
3. Consider a decidable language L . Show that its complement \bar{L} is also decidable.
4. Show that there is a language that is Turing-Recognizable but not Turing-Decidable using the result above.

Sample Solution

1. Let T be the Turing Machine deciding the language $\{\langle D \rangle \mid D \text{ is a DFA with } L(D) = \emptyset\}$ (known from the lecture). We have $L(R) \subseteq L(S) \Leftrightarrow L(R) \setminus L(S) = \emptyset$. Thus we construct a decider for A in the following way.
 A' = " On input $\langle R, S \rangle$ where R, S are regular expression:
 1. Convert R and S into equivalent DFAs (like in the lecture)
 2. Construct a DFA D for the regular language $L(R) \setminus L(S) = \overline{\overline{L(R)} \cup L(S)}$
 3. Run T on input $\langle D \rangle$. Accept iff T accepts."
2. From the the fact, we deduce that $1^* \cap L(G)$ is context free. We construct a TM that decides on B as follows.
 B' = " On input $\langle G \rangle$, where G is a context free grammar :
 1. Construct a CFG C such that $L(C) = 1^* \cap L(G)$
 2. Test whether $L(C) = \phi$ using the decider R for E_{CFG} (cf. lecture).
 3. If R accepts, reject; if R rejects, accept."
3. As the language L is decidable, it has a Turing Machine M that halts on every input string w . So, it either accepts and halts, or rejects and halts. Using this, we can construct a TM M' that does the following :
 1. Run M on input w .
 2. If M accepts, M' rejects.
 3. If M rejects, M' accepts.

This TM M' not only recognizes the language L , but also is decidable, as it halts on every input.

4. Suppose there is a language L that is Turing-Recognizable. Then its complement \bar{L} is not Turing-Recognizable, and this means this language is also not Turing-Decidable, i.e. Undecidable. That derives the fact that this language L is also undecidable.