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Theoretical Computer Science - Bridging Course
Sample Solution Exercise Sheet 11

Due: Tuesday, 11th of July 2023, 12:00 pm

Exercise 1: Construct Formulae *(1+1+1 Points)*

Let $\mathcal{S} = \langle \{x, y, z\}, \emptyset, \emptyset, \{R\} \rangle$ be a signature. Translate the following sentences of first order formula over $\mathcal{S}$ into idiomatic English. Use $R(x, y)$ as statement 'x is a part of y'.

(a) $\exists x \forall y R(x, y)$.

(b) $\exists y \forall x R(x, y)$.

(c) $\forall x \forall y \exists z (R(x, z) \wedge R(y, z))$

Sample Solution

Note that idiomatic English uses, contains, or denotes expressions that are natural to a native speaker. It does not contain variables. it might be said: A native speaker will then read the formulas as

(a) Something is a part of everything.

(b) Something has everything as a part.

(c) For any two things, there is something of which they are both a part.

Exercise 2: FOL: Is it a model? *(2+3+3 Points)*

Consider the following **first order** formulae

$$\begin{aligned}\varphi_1 &:= \forall x R(x, x)\\ \varphi_2 &:= \forall x \forall y\, R(x, y) \rightarrow (\exists z R(x, z) \wedge R(z, y))\\ \varphi_3 &:= \exists x \exists y\, (\neg R(x, y) \wedge \neg R(y, x))\end{aligned}$$

over signature $\mathcal{S}$ where x, y, z are variable symbols and R is a binary predicate. Give an interpretation

(a) I_1 which is a **model** of $\varphi_1 \wedge \varphi_2$.

(b) I_2 which is **no model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

(c) I_3 which is a **model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

Sample Solution

(a) Pick $I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$ where $R^{I_1}(x, y) :\Longleftrightarrow x \leq_{\mathbb{R}} y$.

This is a model because '$\leq_{\mathbb{R}}$' is *reflexive*, therefore fulfills φ_1. Moreover for every $x, y \in \mathbb{R}$ with $x \leq_{\mathbb{R}} y$ we can choose $z := x$, which fulfills $x \leq_{\mathbb{R}} z \land z \leq_{\mathbb{R}} y$. Thus φ_2 is also satisfied.

(b) Let $U = \{1, 2, 3, 4, 5\}$ and *P(U)* be its power set. Pick $I_2 := \langle P(U), \cdot^I \rangle$ where $R^{I_2}(x, y) :\Longleftrightarrow x \subset y$. This is not a model since it doesn't satisfy φ_1, indeed no set is proper subset of itself.

(c) Take two disjoint copies of $\mathbb{R}$ and the standard $\leq_{\mathbb{R}}$ relation on each of them; if x and y are from different copies they are not related in $\mathbb{R}$. Formally let

$$I_3 := \langle \{(a, 1) \mid a \in \mathbb{R}\} \dot{\cup} \{(a, 2) \mid a \in \mathbb{R}\}, \cdot^{I_3} \rangle$$

where $R^{I_3}((a, g), (b, h)) \Leftrightarrow (g = h \text{ and } a \leq_{\mathbb{R}} b)$.

This is a model because $\leq_{\mathbb{R}}$ is *reflexive*, therefore I_3 fulfills φ_1. Furthermore for every two $x = (a, g)$ and $y = (b, h)$ with $R^{I_3}((a, g), (b, h))$, i.e., $g = h$, we can choose $z := (a, g)$ which fulfills $R^{I_3}((a, g), (a, g)) \land R^{I_3}((a, g), (b, h))$. Thus φ_2 is also satisfied. φ_3 is also satisfied, e.g., $(5, 1)$ and $(7, 2)$ are incomparable, i.e., we have neither $R^{I_3}((5, 1), (7, 2))$ nor $R^{I_3}((7, 2), (5, 1))$

Exercise 3: FOL: Entailment *(3+3+3 Points)*

Let φ, ψ be first order formulae over signature $\mathcal{S}$. Similar to propositional logic, in predicate logic we write $\varphi \models \psi$ if every model of φ is also a model for ψ. We write $\varphi \equiv \psi$ if both $\varphi \models \psi$ and $\psi \models \varphi$. A *knowledge base* KB is a set of formulae. A model of KB is model for all formulae in KB. We write $KB \models \varphi$ if all models of KB are models of φ. Show or disprove the following entailments.

(a) $(\exists x\, R(x)) \land (\exists x\, P(x)) \land (\exists x\, T(x)) \models \exists x\, (R(x) \land P(x) \land T(x))$.

(b) $(\forall x \forall y\, f(x, y) \doteq f(y, x)) \land (\forall x\, f(x, \mathbf{c}) \doteq x) \models \forall x\, f(\mathbf{c}, x) \doteq x$.

(c) $(\forall x\, R(x, x)) \,\land\, (\forall x \forall y\, R(x, y) \land R(y, x) \rightarrow x \doteq y) \,\land\, (\forall x \forall y \forall z\, R(x, y) \land R(y, z) \rightarrow R(x, z))$
$\models \forall x \forall y\, R(x, y) \lor R(y, x)$.

Hint: Consider order relations. E.g., $a \leq b$ (a less-equal b) and $a|b$ (a divides b).

Sample Solution

(a) The stated entailment is false (it holds in the other direction though). In order to disprove it, we give a model for the left side which is not a model for the right side.

Let $I = \langle \{a, b\}, \cdot^I \rangle$ with $R^I = \{a\}$, $P^I = \{b\}$, and $T^I = \{a\}$. This makes the left side true since there exists an element $x = a$ that makes $R(x)$ and $T(x)$ true and an element $x = b$ that makes $P(x)$ true (Note the brackets around the three $\exists$ quantifiers which mean that the three elements need not necessarily be the same).

However $R(a) \land P(a) \land T(a) = T \land F \land T = F$ and $R(b) \land P(b) \land T(b) = F \land T \land F = F$ thus the right side is false (there exists no element which makes the three relations' symbols R, P, T true, since we tested all that are in the domain).

(b) The stated entailment holds. We prove this by picking an arbitrary *model* (!) $I = \langle \mathcal{D}, \cdot^I \rangle$ of the left-hand formula. We show that I is a model for the right-hand formula, too. For that purpose let x be an arbitrary element from $\mathcal{D}$.

Since I is a model for the left side we already know $f(x, \mathbf{c}^I) \doteq x$. The first condition in the left formula encodes the commutative property. Since $\mathbf{c}^I$ is also an element from the domain $\mathcal{D}$ we know $f(x, \mathbf{c}^I) = f(\mathbf{c}^I, x)$ and thus $f(\mathbf{c}^I, x) \doteq x$. Since x was arbitrary we have $\forall x\, f(\mathbf{c}^I, x) \doteq x$.

(c) The formula on the left side encodes the properties of an *order relation*. The formula on the right side encodes the property of *totality* of an order, which means that every element is related to (read: can be compared with) every other element. However, in general an order relation does not need to be total (which is called a *partial order*).

The hint proposes two order relations, one of which is total over the domain of integers $\mathbb{Z}^*$ (either '$x \leq y$' or '$x \leq y$' or both) whereas the other is not (it may happen that neither $x|y$ nor $x|y$). Thus the logical entailment is false since with $\mathbb{Z}^*$ and the 'divides'-relation we have a model of the left-hand formula which is no model of the right-hand one (it is not total).

We formalize this as follows. Let $I = \langle \mathbb{Z}^*, \cdot^I \rangle$ with $R^I := \{(x, y) \in \mathbb{Z}^* \mid x \text{ divides } y\}$. Obviously we have the reflexive property since $x \in \mathbb{Z}^*$ divides itself. If $x \in \mathbb{Z}^*$ divides $y \in \mathbb{Z}^*$ and $y \in \mathbb{Z}^*$ divides $z \in \mathbb{Z}^*$ then x also divides z which gives us transitivity. Finally, if x divides y and vice versa then y is multiple of x and vice versa which means that the multiplicand must in both cases be 1, thus both x and y are equal which gives us the antisymmetry property.

This means that I is a model of the left-hand formula. Now consider the two primes $x = 2$ and $y = 3$. By definition of prime numbers neither of the two can divide the other. Thus $\forall x \forall y\, R(x, y) \lor R(y, x)$ is false. Therefore I can be no model of the right-hand formula.