



## Algorithms and Datastructures

### Exercise Sheet 10

#### Exercise 1: Dijkstra's Algorithm

(10 Points)

In the lecture we saw that the runtime of Dijkstra using Fibonacci heaps is  $O(m + n \log n)$ . Is this the actual runtime of the algorithm? Maybe our analysis is just not good enough! We will show that the analysis is indeed tight.

- (a) Argue that any algorithm solving SSSP (Single Source Shortest Paths) must spend at least  $\Omega(m)$  time. An intuitive explanation is sufficient. (1 Point)
- (b) Proof that the Dijkstra algorithm determines shortest paths in a sorted order. For a source node  $v$  and any other two nodes  $u \neq w$  then  $u$  will be marked before  $w$  if  $d(v, u) < d(v, w)$ . Give a formal proof. (4 Points)
- (c) Proof that Dijkstras Algorithm needs  $\Omega(n \log n)$  time if the algorithm is implemented using a comparison based heap. The idea is the following: reduce the problem of sorting  $n$  numbers to the SSSP problem. (Given  $n$  numbers in an array  $A$  create an instance of SSSP.) Give a precise description and a formal proof. (5 Points)

#### Exercise 2: Currency Exchange

(10 Points)

Consider  $n$  currencies  $w_1, \dots, w_n$ . The exchange rates are given in an  $n \times n$ -matrix  $A$  with entries  $a_{ij}$  ( $i, j \in \{1, \dots, n\}$ ). Entry  $a_{ij}$  is the exchange rate from  $w_i$  to  $w_j$ , i.e., for one unit of  $w_i$  one gets  $a_{ij}$  units of  $w_j$ .

Given a currency  $w_{i_0}$ , we want to find out whether there is a sequence  $i_0, i_1, \dots, i_k$  such that we make profit if we exchange one unit of  $w_{i_0}$  to  $w_{i_1}$ , then to  $w_{i_2}$  etc. until  $w_{i_k}$  and then back to  $w_{i_0}$ .

- (a) Translate this problem to a graph problem. That is, define a graph and a property which the graph fulfills if and only if there is a sequence of currencies as described above. (4 Points)
- (b) Give an algorithm that decides in  $\mathcal{O}(n^3)$  time steps whether there is a sequence of currencies as described above. Explain the correctness and runtime. (6 Points)

*Hint: It is  $a \cdot b > 1 \iff -\log a - \log b < 0$*