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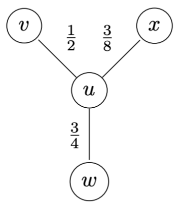
Theory of Distributed Systems
Sample Solution Exercise Sheet 12

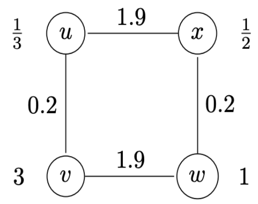
Exercise 1: Selfish Caching

For each of the following caching networks, compute the social optimum, the pure Nash equilibria, the price of anarchy (PoA) as well as the optimistic price of anarchy (OPoA):

1. $d_u = d_v = d_w = d_x = 1$,



2. The demand is written next to each node.



Sample Solution

To be sure that we find every Nash equilibrium, we explicitly write down every best response.

1. **Best-response strategies:**

$$u : \text{cache only if nobody else does. (B1)}$$
$$v : \text{cache if neither } u \text{ nor } x \text{ cache. (B2)}$$
$$w : \text{cache unless } u \text{ caches. (B3)}$$
$$x : \text{cache if neither } u \text{ nor } v \text{ cache. (B4)}$$

Nash equilibrium. If we assume that u plays $Y_u = 1$ (i.e. caches), the system can only be in a NE if $Y_v = Y_w = Y_x = 0$ due to (B1). Since for all v, w, x it is then a best response not to cache, $(1, 0, 0, 0)$ is a Nash equilibrium. If instead $Y_u = 0$, then (B3) implies $Y_w = 1$.

- If $Y_v = 1$, (B2) forces $Y_x = 0$, yielding $(0, 1, 1, 0)$.
- If $Y_v = 0$, (B2) forces $Y_x = 1$, yielding $(0, 0, 1, 1)$.

Hence

$$\text{NE} = \{(1,0,0,0),\ (0,1,1,0),\ (0,0,1,1)\}.$$

Price of Anarchy. The social optimum is achieved at $(1,0,0,0)$:

$$\text{OPT} = \text{cost}(1,0,0,0) = 1 + \tfrac{1}{2} + \tfrac{3}{8} + \tfrac{3}{4} = \tfrac{21}{8}.$$

Since $(1,0,0,0)$ is itself a NE, the optimistic PoA is

$$\text{OPoA} = \frac{\min_{\text{NE}} \text{cost}}{\text{OPT}} = 1.$$

The worst-case PoA comes from $(0,1,1,0)$:

$$\text{PoA} = \frac{\text{cost}(0,1,1,0)}{\text{OPT}} = \frac{\frac{1}{2} + 1 + 1 + \frac{7}{8}}{\frac{21}{8}} = \frac{9/8}{21/8} = \tfrac{9}{7} \approx 1.286.$$

2. **Best-response strategies:**

$$\begin{aligned} u &: \text{cache only if nobody else does.} && (B1)\\ v &: \text{cache unless } u \text{ caches.} && (B2)\\ w &: \text{cache unless } x \text{ caches.} && (B3)\\ x &: \text{cache if neither } u \text{ nor } w \text{ cache.} && (B4) \end{aligned}$$

Nash equilibrium. If we assume that u plays $Y_u = 1$ (i.e. u caches) the system can only be in a NE if $Y_v = Y_w = Y_x = 0$ due to (B1). However, $Y_x = 0$ implies $Y_w = 1$ by (B3), so there can be no NE with $Y_u = 1$. Thus in any NE we must have $Y_u = 0$, and hence $Y_v = 1$ by (B2).

- If $Y_w = 1$, then (B3) implies $Y_x = 0$. This does not violate (B4), and so $(Y_u, Y_v, Y_w, Y_x) = (0,1,1,0)$ is a Nash equilibrium.
- If $Y_w = 0$, then (B4) implies $Y_x = 1$, and so $(0,1,0,1)$ is also a Nash equilibrium.

Price of anarchy. The social optimum is achieved at $(0,1,1,0)$, namely

$$\text{OPT} = \text{cost}(0110) = \tfrac{1}{3} \cdot 0.2 + 1 + 1 + \tfrac{1}{2} \cdot 0.2 = 2.\overline{16}.$$

Since (0110) is also a NE, the *optimistic* price of anarchy is 1. The *worst-case* price of anarchy is

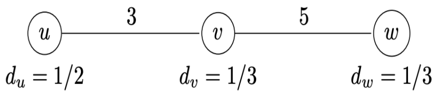
$$\text{PoA} = \frac{\text{cost}(0101)}{\text{OPT}} = \frac{\frac{1}{3} \cdot 0.2 + 1 + 0.2 + 1}{2.\overline{16}} = \frac{68}{65} \approx 1.046.$$

Exercise 2: Selfish Caching with variable caching cost

The selfish caching model introduced in the lecture assumed that every peer incurs the same caching cost. However, this is a simplification of reality: a peer with little storage space could experience a much higher caching cost than a peer who has terabytes of free disk space. In this exercise, we omit the simplifying assumption and allow variable caching costs α_i for node i.
What are the Nash equilibria in the following caching networks given that

1. $\alpha_u = 1, \quad \alpha_v = 2, \quad \alpha_w = 2$?
2. $\alpha_u = 3, \quad \alpha_v = \frac{3}{2}, \quad \alpha_w = 3$?

Does any of the above instances admit a dominant-strategy profile? What is the Price of Anarchy in each case?



Sample Solution

We define $D_i = \{j : c_{i \leftarrow j} < \alpha_i\}$ as the set of nodes that "cover" node i. A profile $Y \in \{0,1\}^3$ is a Nash equilibrium if and only if for each node i:

- If $Y_i = 1$ then $Y_j = 0$ for all $j \in D_i$.
- If $Y_i = 0$ then there exists $j \in D_i$ with $Y_j = 1$.

1. $D_u = \emptyset,\ D_v = \{u, w\},\ D_w = \{u\}$.

$D_u = \emptyset$ forces $Y_u = 1$.

- Then $Y_v = 0$, and hence $Y_w = 1$.

Thus

$$\text{NE} = \{(1,0,1)\},$$

which is also the social optimum, so PoA = 1.

2. $D_u = \{v\},\ D_v = \{u\},\ D_w = \{u, v\}$. Case-by-case:

- If $Y_u = 1$, then $Y_v = Y_w = 0$, giving profile $(1,0,0)$.
- If $Y_u = 0$, then $Y_v = 1$, which forces $Y_w = 0$, giving $(0,1,0)$.

Hence

$$\text{NE} = \{(1,0,0),\ (0,1,0)\}.$$

Their costs are

$$\text{cost}(1,0,0) = 3 + 1 + \frac{8}{3}, \quad \text{cost}(0,1,0) = \frac{3}{2} + \frac{3}{2} + \frac{5}{3},$$

so

$$\text{PoA} = \frac{\max\{\text{cost}(1,0,0),\ \text{cost}(0,1,0)\}}{\min\{\text{cost}(1,0,0),\ \text{cost}(0,1,0)\}} = \frac{\frac{40}{3}}{\frac{28}{3}} = \frac{40}{28} \approx 1.43.$$

Dominant strategies Every dominant strategy profile is also a Nash equilibrium. Hence we only have to check the computed NEs to see whether they consist solely of dominant strategies.
Let us consider **game** (1). Since every dominant strategy profile is also a Nash equilibrium, it suffices to look at the NE. The game has no dominant strategy profile. In particular, profile (101) is not a dominant strategy profile in game 1: although $Y_u = 1$ is the dominant strategy for u, the choices $Y_v = 0$ and $Y_w = 1$ are not dominant for v and w. (Indeed, if $Y_v = 1$, then the best response of w would be $Y_w = 0$.) **Game 2:** Since the decision of node u whether to cache depends on the decision of node v, it is not a dominant strategy. Therefore this game also has no dominant strategy profile.

Exercise 3: Matching Pennies

Tobias and Stephan like to gamble, and came up with the following game: Each of them secretly turns a penny to heads or tails. Then they reveal their choices simultaneously. If the pennies match Tobias gets both pennies, otherwise Stephan gets them.
Write down this 2-player game as a bi-matrix, and compute its (mixed) Nash equilibria!

Sample Solution

The bi-matrix of the zero-sum game with Tobias (row) and Stephan (column) is

	H	T
H	$(+1, -1)$	$(-1, +1)$
T	$(-1, +1)$	$(+1, -1)$

There is no pure-strategy Nash equilibrium. Let Tobias play H with probability p and T with $1-p$, and Stephan play H with probability q and T with $1-q$. Their expected payoffs are

$$\begin{aligned}
\Gamma_T(p,q) &= p\big(q\cdot(+1)+(1-q)\cdot(-1)\big)+(1-p)\big(q\cdot(-1)+(1-q)\cdot(+1)\big)\\
&= (q-(1-q))\,p+\big(-(q-(1-q))\big)(1-p)\\
&= (2q-1)\,p-(2q-1)(1-p)=(4q-2)\,p+1-2q,\\
\Gamma_S(p,q) &= q\big(p\cdot(-1)+(1-p)\cdot(+1)\big)+(1-q)\big(p\cdot(+1)+(1-p)\cdot(-1)\big)\\
&= (-(p-(1-p)))\,q+(p-(1-p))(1-q)\\
&= (2p-1)(1-q)-(2p-1)\,q=(2-4p)\,q+2p-1.
\end{aligned}$$

For Stephan, $\Gamma_S(p,q)$ is constant in q exactly when $p=\frac{1}{2}$; for Tobias, $\Gamma_T(p,q)$ is constant in p exactly when $q=\frac{1}{2}$. Thus the unique mixed-strategy Nash equilibrium is

$$(p^*,q^*)=\left(\tfrac{1}{2},\tfrac{1}{2}\right).$$