



Theoretical Computer Science - Bridging Course

Exercise Sheet 9

Due: Tuesday, 1st of July 2025, 12:00 pm

Exercise 1: Class $\mathcal{NPC}$

(3+4+5 Points)

Recall: Let L_1, L_2 be languages (problems) over alphabets Σ_1, Σ_2 . Then $L_1 \leq_p L_2$ (L_1 is polynomially reducible to L_2), iff a function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ exists, that can be calculated in polynomial time and

$$\forall s \in \Sigma_1^* : s \in L_1 \iff f(s) \in L_2.$$

Language L is called $\mathcal{NP}$ -hard, if *all* languages $L' \in \mathcal{NP}$ are polynomially reducible to L , i.e.

$$L \text{ is } \mathcal{NP}\text{-hard} \iff \forall L' \in \mathcal{NP} : L' \leq_p L.$$

The reduction relation ' $\leq_p$ ' is transitive ($L_1 \leq_p L_2$ and $L_2 \leq_p L_3 \Rightarrow L_1 \leq_p L_3$). Therefore, in order to show that L is $\mathcal{NP}$ -hard, it suffices to reduce a known $\mathcal{NP}$ -hard problem $\tilde{L}$ to L , i.e. $\tilde{L} \leq_p L$.

Finally a language is called $\mathcal{NP}$ -complete ($\Leftrightarrow L \in \mathcal{NPC}$), if

1. $L \in \mathcal{NP}$ and
2. L is $\mathcal{NP}$ -hard.

Now, show that the following problems are in $\mathcal{NPC}$

- (a) $\text{INDEPENDENTSET} := \{ \langle G, k \rangle \mid G \text{ has an independent set of size at least } k \}$, where an independent set is a subset of nodes of G such that no two nodes in the subset share an edge in G .
- (b) $\text{CLIQUE} := \{ \langle G, k \rangle \mid G \text{ has a clique of size at least } k \}$.
- (c) $\text{HITTINGSET} := \{ \langle \mathcal{U}, S, k \rangle \mid \text{universe } \mathcal{U} \text{ has subset of size at most } k \text{ that hits all sets in } S \subseteq 2^{\mathcal{U}} \}$.

(Bonus) $\text{DOMINATINGSET} := \{ \langle G, k \rangle \mid \text{Graph } G \text{ has a dominating set of size at most } k \}$, where a dominating set is a subset of nodes of G such that every node in G is in the subset or has a neighbor in the subset.

Hint: For all four parts, use the fact that $\text{VERTEXCOVER} := \{ \langle G, k \rangle \mid \text{Graph } G \text{ has a vertex cover of size at most } k \} \in \mathcal{NPC}$, where a vertex cover is a subset of nodes of G such that every edge of G is incident to a node in the subset.

For the poly. transformation ($\leq_p$) you have to describe an algorithm (with poly. run-time!) that transforms:

For part (a), an instance $\langle G, k \rangle$ of VERTEXCOVER into an instance $\langle G', k' \rangle$ of INDEPENDENTSET s.t. a vertex cover of size $\leq k$ in G becomes an independent set of G' of size $\geq k'$ vice versa(!)

For part (b), an instance $\langle G, k \rangle$ of VERTEXCOVER into an instance $\langle G', k' \rangle$ of CLIQUE s.t. a vertex cover of size $\leq k$ in G becomes a clique of G' of size $\geq k'$ vice versa(!)

For part (c), an instance $\langle G, k \rangle$ of VERTEXCOVER into an instance $\langle \mathcal{U}, S, k' \rangle$ of HITTINGSET, s.t. a vertex cover of size $\leq k$ in G becomes a hitting set of $\mathcal{U}$ of size $\leq k'$ for S and vice versa(!).

For the bonus, transform an instance $\langle G, k \rangle$ of VERTEXCOVER into an instance $\langle G', k' \rangle$ of DOMINATINGSET s.t. a vertex cover of size $\leq k$ in G becomes a dominating set of G' of size $\leq k'$ vice versa(!)

Note that a dominating set is not necessarily a vertex cover ($G = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\})$ has the dominating set $\{v_1, v_4\}$ which is not a vertex cover). Also a vertex cover is not necessarily a dominating set (consider isolated nodes).

Exercise 2: Complexity Classes: Big Picture

(2+3+3 Points)

- (a) Why is $\mathcal{P} \subseteq \mathcal{NP}$?
- (b) Show that $\mathcal{P} \cap \mathcal{NPC} = \emptyset$ if $\mathcal{P} \neq \mathcal{NP}$.
Hint: Assume that there exists a $L \in \mathcal{P} \cap \mathcal{NPC}$ and derive a contradiction to $\mathcal{P} \neq \mathcal{NP}$.
- (c) Give a Venn Diagram showing the sets $\mathcal{P}, \mathcal{NP}, \mathcal{NPC}$ for both cases $\mathcal{P} \neq \mathcal{NP}$ and $\mathcal{P} = \mathcal{NP}$.
Remark: Use the results of (a) and (b) even if you did not succeed in proving those.