



# Theoretical Computer Science - Bridging Course

## Exercise Sheet 10

Due: Tuesday, 8th of July 2025, 12:00 pm

### Exercise 1: Propositional Logic: Basic Terms (1+1+1+1 Points)

Let  $\Sigma := \{p, q, r\}$  be a set of atoms. An interpretation  $I : \Sigma \rightarrow \{T, F\}$  maps every atom to either true or false. Inductively, an interpretation  $I$  can be extended to composite formulae  $\varphi$  over  $\Sigma$  (cf. lecture). We write  $I \models \varphi$  if  $\varphi$  evaluates to  $T$  (true) under  $I$ . In case  $I \models \varphi$ ,  $I$  is called a *model* for  $\varphi$ .

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a)  $\varphi_1 = (p \wedge \neg q) \vee (\neg p \vee q)$

(b)  $\varphi_2 = (\neg p \wedge (\neg p \vee q)) \leftrightarrow (p \vee \neg q)$

(c)  $\varphi_3 = (p \wedge \neg q) \rightarrow \neg(p \wedge q)$

(d)  $\varphi_4 = (p \wedge q) \rightarrow (p \vee r)$

*Remark:*  $a \rightarrow b \equiv \neg a \vee b$ ,  $a \leftrightarrow b \equiv (a \rightarrow b) \wedge (b \rightarrow a)$ ,  $a \not\rightarrow b \equiv \neg(a \rightarrow b)$ .

### Exercise 2: CNF and DNF (2+2 Points)

(a) Convert  $\varphi_1 := (p \rightarrow q) \rightarrow (\neg r \wedge q)$  into Conjunctive Normal Form (CNF).

(b) Convert  $\varphi_2 := \neg((\neg p \rightarrow \neg q) \wedge \neg r)$  into Disjunctive Normal Form (DNF).

*Remark:* Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step.

### Exercise 3: Logical Entailment (3+3 Points)

A *knowledge base*  $KB$  is a set of formulae over a given set of atoms  $\Sigma$ . An interpretation  $I$  of  $\Sigma$  is called a *model* of  $KB$ , if it is a model for *all* formulae in  $KB$ . A knowledge base  $KB$  *entails* a formula  $\varphi$  (we write  $KB \models \varphi$ ), if *all* models of  $KB$  are also models of  $\varphi$ .

Let  $KB := \{p \vee q, \neg r \vee p\}$ . Show or disprove that  $KB$  logically entails the following formulae.

(a)  $\varphi_1 := (p \wedge q) \vee \neg(\neg r \vee p)$

(b)  $\varphi_2 := (q \leftrightarrow r) \rightarrow p$

## Exercise 4: Inference Rules and Calculi

(3+3 Points)

Let  $\varphi_1, \dots, \varphi_n, \psi$  be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \dots, \varphi_n}{\psi}$$

means that if  $\varphi_1, \dots, \varphi_n$  are 'considered true', then  $\psi$  is 'considered true' as well ( $n = 0$  is the special case of an axiom). A (propositional) *calculus* **C** is described by a *set* of inference rules.

Given a formula  $\psi$  and knowledge base  $KB := \{\varphi_1, \dots, \varphi_n\}$  (where  $\varphi_1, \dots, \varphi_n$  are formulae) we write  $KB \vdash_{\mathbf{C}} \psi$  if  $\psi$  can be derived from  $KB$  by starting from a subset of  $KB$  and repeatedly applying inference rules from the calculus **C** to 'generate' new formulae until  $\psi$  is obtained.

Consider the following two calculi, defined by their inference rules ( $\varphi, \psi, \chi$  are arbitrary formulae).

$$\begin{aligned} \mathbf{C}_1 : & \quad \frac{\varphi \rightarrow \psi, \psi \rightarrow \chi}{\varphi \rightarrow \chi}, \frac{\neg \varphi \rightarrow \psi}{\neg \psi \rightarrow \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \rightarrow \psi, \psi \rightarrow \varphi} \\ \mathbf{C}_2 : & \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi}, \frac{\varphi \wedge \psi}{\varphi, \psi}, \frac{(\varphi \wedge \psi) \rightarrow \chi}{\varphi \rightarrow (\psi \rightarrow \chi)} \end{aligned}$$

Using the respective calculus, show the following derivations (document your steps).

- (a)  $\{p \leftrightarrow \neg r, \neg q \rightarrow r\} \vdash_{\mathbf{C}_1} p \rightarrow q$
- (b)  $\{p \wedge q, p \rightarrow r, (q \wedge r) \rightarrow s\} \vdash_{\mathbf{C}_2} s$

*Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.*