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Theoretical Computer Science - Bridging Course
Sample Solution Exercise Sheet 1

Due: Tuesday, 29th of April 2025, 12:00 pm

Exercise 1: Miscellaneous Mathematical Proofs *(3+1+1+2+1 Points)*

1. Let $S(n) = \sum_{i=1}^{n} i$ be the sum of the first n natural numbers and $C(n) = \sum_{i=1}^{n} i^3$ be the sum of the first n cubes. Use mathematical induction to prove the following interesting conclusion: $C(n) = S^2(n)$ for every integer $n \geq 0$.

2. Let A, B, and C be subsets of U. Which of the following statements is true? Justify.
   - If $A \cap B = A \cap C$, then $B = C$.
   - If $A \cup B = A \cup C$, then $B = C$.
   - $\overline{A \cup B} = \overline{A} \cap \overline{B}$, where $\overline{A}$ is the compliment of A.

3. Let $A_1, A_2, ..., A_k$ be nonempty subsets of U, where k is any positive integer. Prove that there exists a nonempty subset $A \subseteq U$ such that $A \cap A_i \neq \phi$, for all $i \in \{1, 2, ..., k\}$.

Sample Solution

1. Base case: for $n = 1$, $1^3 = (1)^2$ is true.

   Induction step: for each $k \geq 1$, we assume that the statement holds true for k i.e. $C(k) = S^2(k)$ (induction hypothesis IH). Now, we need to prove that the statement holds true for $k+1$ i.e. we want to show that $C(k+1) = S^2(k+1)$.
   Indeed first, we recall that $S(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, hence $S^2(k+1) = \big(\frac{(k+1)(k+2)}{2}\big)^2 = \frac{(k+1)^2(k+2)^2}{4}$.
   Next, we have that $C(k+1) = \sum_{i=1}^{k} i^3 + (k+1)^3 = C(k) + (k+1)^3 \overset{\text{IH}}{=} S^2(k) + (k+1)^3 = \big(\frac{k(k+1)}{2}\big)^2 + (k+1)^3 = \big(\frac{k^2(k+1)^2}{4}\big) + (k+1)^3 = \frac{(k+1)^2}{4}(k^2 + 4k + 4) = \frac{(k+1)^2}{4}(k+2)^2 = S^2(k+1)$.
   Hence, the statement holds true for $k+1$, which ends our induction proof.

2.
   - False. We give a counterexample: take $A = \{1, 2, 3\}$, $B = \{1, 4\}$ and $C = \{1, 5\}$, hence $A \cap B = A \cap C$ and $B \neq C$.
   - False. We give a counterexample: take $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 3\}$, hence $A \cup B = A \cup C$ and $B \neq C$.
   - (De Morgan's law). Indeed,

     $$x \in \overline{A \cup B} \iff x \notin A \cup B \iff x \notin A \text{ and } x \notin B \iff x \in \overline{A} \text{ and } x \in \overline{B} \iff x \in \overline{A} \cap \overline{B}$$

     hence, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

3. We construct A by choosing one element from each A_i, for all $i \in \{1, 2, ..., k\}$.

Exercise 2: Graphs (Part 1) *(3+3 Points)*

A *simple graph* is a graph without self loops, i.e., every edge of the graph is an edge between two distinct nodes. The degree $d(v)$ of a node $v \in V$ in an undirected graph $G = (V, E)$ is the number of its neighbors, i.e, $d(v) = |\{u \in V \mid \{v, u\} \in E\}|$. Let $m \geq 0$ denote the number of edges in graph G.

1. Prove the handshaking lemma i.e. $\sum_{v \in V} d(v) = 2m$ via mathematical induction on m *for any simple graph* $G = (V, E)$.

2. Show that every simple graph with an odd number of nodes contains a node with even degree.

Sample Solution

1. We prove the handshaking lemma by mathematical induction on m.

   Base step: let $G = (V, E)$ be a graph where $|V| = n$ and $|E| = m = 0$. Notice that G is the empty graph on n nodes, hence $\sum_{v \in V} d(v) = 0$, thus the handshaking lemma is true on G.

   Induction step: for each k, we assume that the statement holds true for k i.e. $\sum_{v \in V} d(v) = 2k$ for any graph $G = (V, E)$ where $|V| = n$ and $|E| = k$ (induction hypothesis IH).
   Now, we need to prove that the statement holds true for $k + 1$ i.e. we want to show that $\sum_{v \in V} d(v) = 2(k + 1)$ for any $G = (V, E)$ where $|V| = n$ and $|E| = k + 1$.

   Indeed, first we consider a graph $G = (V, E)$ where $|V| = n$ and $|E| = k + 1$. Let $\{u, v\}$ be an edge in G. Let $G' = (V, E')$ where $E' = E \backslash \{x, y\}$ i.e. G' is the graph obtained after removing an edge $\{x, y\}$ from G. Note that we denote by $d_G(v), d_{G'}(v)$ the degree of node v in G and G' respectively.

   First we notice that G' has k edges, hence by IH $\sum_{v \in V} d_{G'}(v) = 2k$.

   Moreover, $\sum_{v \in V} d_{G'}(v) = \sum_{v \in V \backslash \{x,y\}} d_{G'}(v) + d_{G'}(x) + d_{G'}(y) = \sum_{v \in V \backslash \{x,y\}} d_G(v) + (d_G(x) - 1) + (d_G(y) - 1) = \sum_{v \in V \backslash \{x,y\}} d_G(v) + d_G(x) + d_G(y) - 2 = \sum_{v \in V} d_G(v) - 2$.

   Thus $\sum_{v \in V} d_G(v) = \sum_{v \in V} d_{G'}(v) + 2 \overset{\text{IH}}{=} 2k + 2 = 2(k + 1)$

   Hence, the statement holds true for $k + 1$, which ends our induction proof.

2. Let $G = (V, E)$ be a graph. We argue by contradiction. Assume that $\forall v \in V$, $d(v)$ is odd. Now since G has odd number of nodes, we notice that $\sum_{v \in V} d(v)$ is the sum of an odd number of odd numbers, which is odd. But by the handshaking lemma $\sum_{v \in V} d(v)$ must be even. This is a contradiction. Thus our assumption must have been false and hence there must exist a node in G with even degree.

Exercise 3: Graphs (Part 2) *(2+4 Points)*

A graph $G = (V, E)$ is said to be *connected* if for every pair of vertices $u, v \in V$ such that $u \neq v$ there exists a path in G connecting u to v.

1. Prove that if G is connected, then for any two nonempty subsets V_1 and V_2 of V such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$, there exists an edge joining a vertex in V_1 to a vertex in V_2.

2. Let G be a simple, connected graph and P be a path of the longest length ℓ in G. Show that if the two ends of P are adjacent, then $V = V(P)$, where $V(P)$ is the set of vertices of P.
   Hint: Try to argue by contradiction.

Sample Solution

Definition: a family of sets $V_1, V_2, ..., V_k$, where k is some positive integer is called a *partition* of V if and only if all of the following conditions hold:

- For all $i \in \{1, 2..., k\}$, V_i is a nonempty subset of V
- $\bigcup_{i=1}^{k} V_i = V$
- $V_i \cap V_j = \phi$ for all $i, j \in \{1, 2..., k\}$ such that $i \neq j$

Intuitively you can think of a partition of a set V as a set of non-empty subsets of V such that every element $x \in V$ is in exactly one of these subsets.

1. Let V_1 and V_2 be any two non empty subsets of V such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$ (i.e. V_1 and V_2 is a partition of the vertex set V). Let $u \in V_1$ and $v \in V_2$. Since G is connected, there exists a path in G joining u to v. For this to happen, there must then exist an edge joining some vertex in V_1 to some other vertex in V_2, which ends our proof.

2. *Notations and definitions:* A path P on n vertices say $\{v_1, v_2, ..., v_n\}$ is a graph whose set of edges is $\{\{v_i, v_{i+1}\}; 1 \leq i \leq n-1\}$ and to describe it we write $P = v_1 v_2 ... v_n$.
   Let v_i and v_j be any two vertices of P, where $1 \leq i \leq j \leq n$, then we denote by $P_{[v_i, v_j]} = v_i v_{i+1} ... v_j$ the subpath of P with ends v_i and v_j.

   Solution: We argue by contradiction. Suppose $V \neq V(P)$, where we define $V(P) := \{v_1, v_2, ..., v_{\ell+1}\}$, then there exists at least one vertex in V that is not in $V(P)$. Hence, we can define $V_1 := V \backslash V(P) \neq \phi$ and $V_2 := V(P) \neq \phi$. Notice that V_1 and V_2 from a partition of V. Moreover since G is connected, by the previous part we deduce that there exists an edge joining a vertex in V_1 (call it x) to a vertex v_k in $V_2 = V(P)$, where $1 \leq k \leq \ell + 1$. Let $P = v_1 v_2 ... v_{\ell+1}$ and $e = \{x, v_k\}$. Since the two ends of P are adjacent i.e. $\{v_1, v_{\ell+1}\} \in E$, we can define another path $P' = x v_k P_{[v_{k+1}, v_\ell]} v_{\ell+1} v_1 P_{[v_2, v_{k-1}]}$. Notice that P' is a path in G of length $\ell + 1$, which is a contradiction. Hence, our supposition is incorrect. Thus, $V = V(P)$.