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Theoretical Computer Science - Bridging Course
Sample Solution Exercise Sheet 3

Due: Tuesday, 13th of May 2025, 12:00 pm

Exercise 1: REs *(2+2+2+2 Points)*

(a) Let $\Sigma = \{a, b\}$. Let L_1 be the language defined by the regular expression $a^*b^*a^*$ and L_2 the language defined by a^*b^*b. Draw a DFA and NFA for L_1 and L_2.

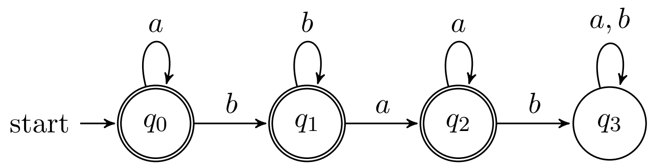
(b) Let $\Sigma = \{a, b, c\}$. What language does the following regular expression describe $((a \cup c)^*b(a \cup c)^*b(a \cup c)^*b(a \cup c)^*)^*$? Also, draw a NFA for $((a \cup c)^*b)^*$.

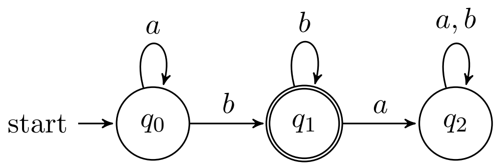
(c) Let $\Sigma = \{a, b\}$. Provide a regular expression that recognizes the following two languages.

- Let language L_3 contain all strings in which at least one of the symbols a or b occurs an even number of times.
- Let language L_4 contain all strings of length at least 2 such that a and b are alternating.

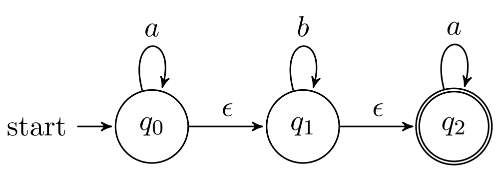
Sample Solution

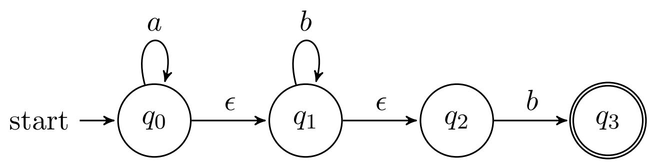
1. The DFA for L_1 and L_2 are respectively as follows:





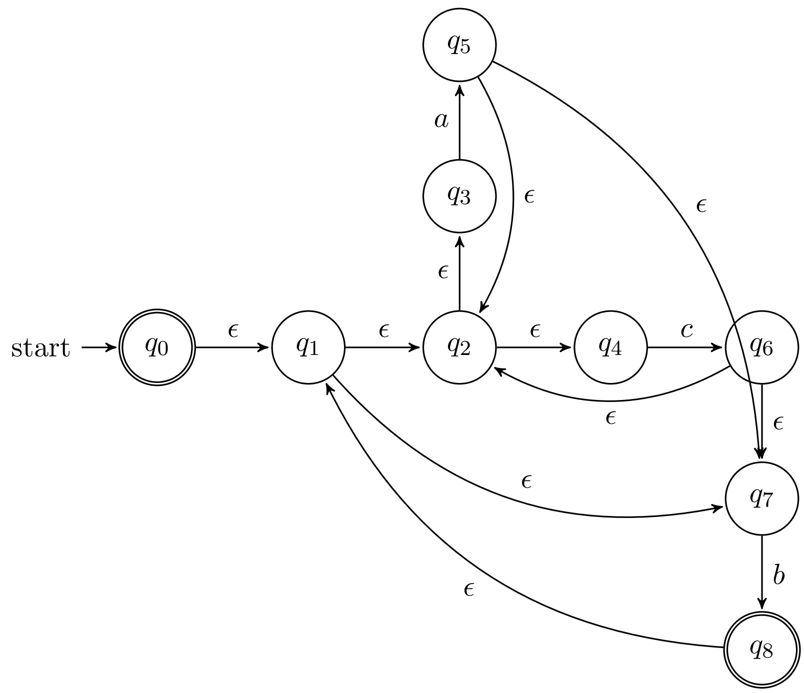
The NFA for L_1 and L_2 are respectively as follows:





2. $L = \{w \in \Sigma^* |$ the number of $b's$ is a non zero multiple of 3 or w is ϵ $\}$.

   The NFA of $((a \cup c)^* b)^*$ is as follows.



3. $a^* \cup b^* \cup (b^*ab^*ab^*)^* \cup (a^*ba^*ba^*)^*$

4. $a(ba)^*b \cup a(ba)^*ba \cup b(ab)^*a \cup b(ab)^*ab$

Exercise 2: Limits of the Pumping Lemma *(1+3 Points)*

Consider the language $L = \{c^m a^n b^n \mid m, n \geq 0\} \cup \{a, b\}^*$ over the alphabet set $\Sigma = \{a, b, c\}$.

(a) Describe in words (not using the pumping lemma), why L can not be a regular language.

(b) Show that, while the property described in the Pumping Lemma is a necessary condition for regularity, it is *not* sufficient for regularity.

   Hint: Use L as counter example, i.e., show that it can be 'pumped' (in the sense of the pumping lemma), but is still not regular.

Sample Solution

Note: The definition of Kleene star's closure is a set of all strings (including empty string ϵ) obtained from the symbols in the alphabet set Σ. It is denoted as Σ^*. Remember, that the definition of a string here is a finite sequence of symbols from an alphabet set. So Kleene's closure contains (finite length) strings and the cardinality of it is infnite, as it could have unbd number of elements.

(a) For recognizing that a word has the same number of a's and b's, a DFA would have to count the number of appearances of these characters, requiring at least one state for each appearance. But as the number of appearances can be arbitrary large, the automaton would need an ∞ number of states. Thus L can not be a regular language.

(b) The goal is to show that L, athough it is *not* a regular language, has the property described in the Pumping Lemma. We will do this in the following.

   For the pumping length, we choose an arbitrary $p \geq 1$. Now, the goal is to show: for every word w in L of length at least p, there exists a composition $w = xyz$ satisfying the three properties from the lemma:

1) for all integers $i \geq 0$, it holds: $xy^iz \in L$
2) $|y| \geq 1$
3) $|xy| \leq p$

If $w \in \{a, b\}^*$, it is clear (since we will have w starting with either a or b, so we can choose $x = \epsilon$, $y = a$ (or $y = b$ depending on what w starts with), and z the rest of the string to be a composition of w. This composition of w will have properties 1, 2, 3 satisfied. And *besides* $\{a, b\}^*$ itself is a regular language so it must satisfy the property described by the Pumping Lemma).
Else, $w = c^m a^n b^n$ with $m \geq 1$. We can choose $x = \epsilon$, $y = c$ and $z = c^{m-1}a^n b^n$ as a composition of w that will have properties 1, 2, 3 satisfied.
Note that by doing this, we would have checked all words w in L of length at least p.

Take away: If we want to show that a language L is regular, then it is not sufficient to show that L has the property described in the Pumping Lemma. So, we can't really use the Pumping Lemma if we want to prove a certain language regular (we can instead try to find a DFA, NFA, or a regular expression for L).

Exercise 3: Proving Non-regularity *(2+3 Points)*

Use the Pumping Lemma to show that the following languages over the alphabet set $\Sigma = \{a, b, c\}$ are not regular.

(a) $L := \{a^n c b^{n+2} \mid n \geq 0\}$.

(b) $L = \{a^m \mid m = n^2 \text{ for some } n \geq 0\}$.

Bonus: $L = \{a^n bwa^n | n \geq 1 \text{ and } w \in \Sigma^*\}$.

Sample Solution

In both parts, we are going to **assume** that L is regular. This means that the property from the Pumping Lemma should hold **true** for L (i.e. there exists a pumping length $p \geq 1$ such that for every string $w \in L$ where $|w| \geq p$, we can break w into three strings $w = xyz$ such that properties 1, 2, and 3 hold true). **But**, we will show that in fact this property will **not** hold **true** for L (i.e. we will show that: for this pumping length $p \geq 1$, there exists a (bad) word $w \in L$ of length at least p such that for every composition of $w = xyz$ that satisfies properties 2 and 3, it turns out property 1 is never satisfied). This will then give us a **contradiction** to the Pumping Lemma, hence, our initial assumption must have been wrong. Therefore, L is not regular.

(a) Assume L is a regular language. This means that the property from the Pumping Lemma should hold true for L.
Now, let $p \geq 1$ be any pumping length. Consider the string $w = a^p c b^{p+2} \in L$. Consider any composition $w = xyz$ with $|y| \geq 1$ and $|xy| \leq p$ (conditions 2 and 3 of the Pumping Lemma). Based on the Pumping Lemma, for all $\ell \geq 0$, $xy^\ell z$ must also be in L. Therefore, xy^3z must be in L, but since y only consists of as, then xy^3z contains at least as many as as bs, which means that it is not in L, in contrast to conidition 1 of the Pumping Lemma, thus a contradiction to the Pumping Lemma. Therefore, L is not regular.

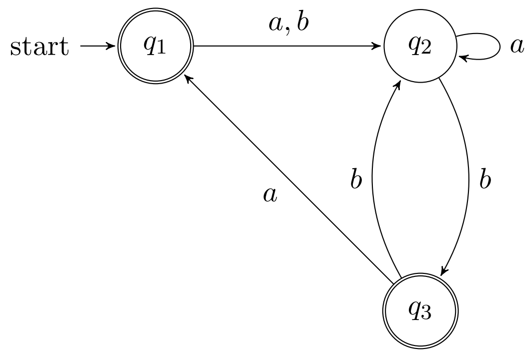
(b) Assume L is a regular language. This means that the property from the Pumping Lemma should hold true for L.
Now, let $p \geq 1$ be any pumping length. Consider a string $w = a^{p^2} \in L$. Consider any composition $w = xyz$ with $|y| \geq 1$ and $|xy| \leq p$ (conditions 2 and 3 of the Pumping Lemma). In particular, we have $|y| \leq p$ and with $|xyz| = p^2$, so we get $|xy^2z| = |xyz| + |y| = p^2 + p \leq p^2 + 2p + 1 = (p+1)^2$. It follows that $|xy^2z| < (p+1)^2$. On the other hand, because of $|y| \geq 1$ we have $|xy^2z| > |xyz| = p^2$.

So $|xy^2z|$ lies strictly between the consecutive squares of integers $p, p+1$. Thus $xy^2z \notin L$, but based on the Pumping Lemma (condition 1), xy^2z must also be in L, thus a contradiction to the Pumping Lemma. Therefore, L is not regular.

Bonus solution: Assume L is a regular language. This means that the property from the Pumping Lemma should hold true for L.
Now, let $p \geq 1$ be any pumping length. Consider a string $w = a^p b a^p \in L$. Consider any composition $w = xyz$ with $|y| \geq 1$ and $|xy| \leq p$. Then we can let $x = a^{p-q-r}, y = a^q, z = a^r b a^p$, for some $q \geq 1$ and $r \geq 0$. Based on the Pumping Lemma, for all $\ell \geq 0$, $xy^\ell z$ must also be in L. Therefore, xy^2z must be in L, but $xy^2z = a^{p+q} b a^p \notin L$, a contradiction to the Pumping Lemma. Therefore, L is not regular.

Exercise 4: NFA-GNFA-RE *(3 Points)*

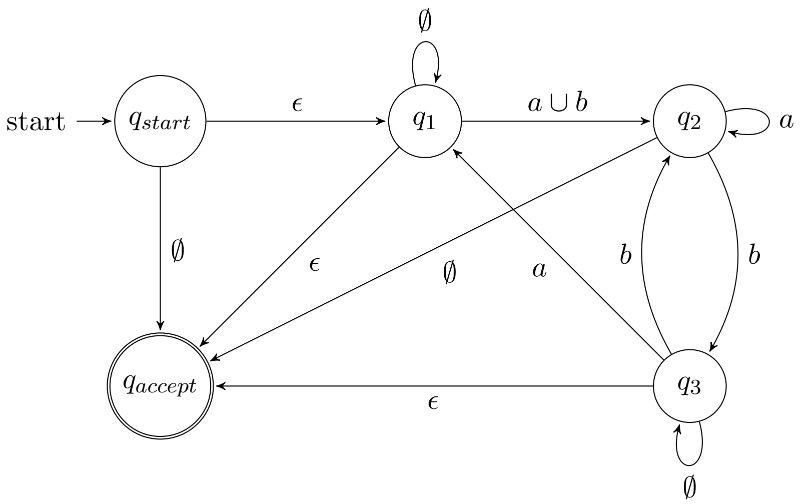
Consider the following NFA:



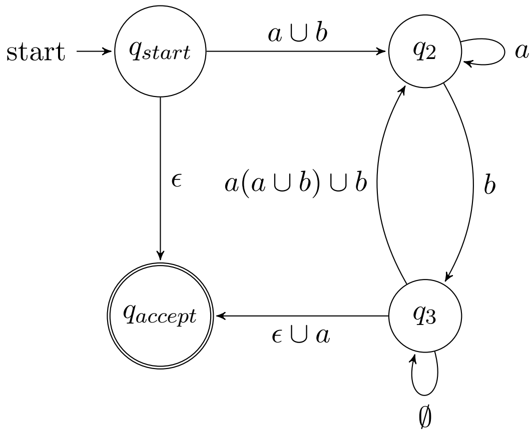
Give the regular expression defining the language recognized by this NFA by converting it *stepwise* into an equivalent GNFA with only two nodes. Document your steps.

Sample Solution

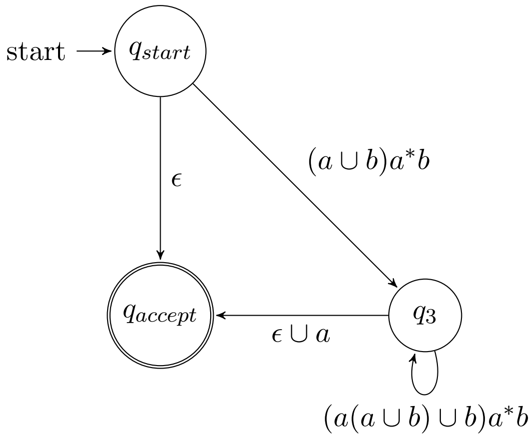
1) Add a new start and accepting state, connect them with ϵ transitions from/to the previous start/accept states, replace multiple labels with unions, add transitions with $\emptyset$ when not present in the original DFA (for a better readability, some edges with label $\emptyset$ are left out in the following diagram):



2) Rip off q_1:



3) Rip off q_2:



4) Rip off q_3: