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Theoretical Computer Science - Bridging Course
Sample Solution Exercise Sheet 4

Due: Tuesday, 20th of May 2025, 12:00 pm

Exercise 1: CFGs *(10 Points)*

Give a context free grammar for each of the following languages.

- $L_1 = \{a^i b^j | 0 < i \leq j\}$
- $L_2 = \{a^{2n} b^n \mid n > 0\}$
- $L_3 = \{a^* w c^k | w \in \{a, b\}^*, \text{ and } k \text{ is the number of } a\text{'s in } w\}$
- $L_4 = \{a^k b^{3k} | k \geq 0\}$
- $L_1 \circ L_2$
- $L_1 \cup L_2$

Sample Solution

Let S_1, S_2, S_3, and S_4 be the respective start variables of the first four grammars.

- $$\begin{aligned} S_1 &\to aS_1b \mid B \\ B &\to Bb \mid ab \end{aligned}$$

- $$S_2 \to aaS_2b \mid aab$$

- $$\begin{aligned} &S_3 \to AW \\ &A \to aA \mid \epsilon \\ &W \to BaBCc \mid B \mid \epsilon \\ &B \to Bb \mid \epsilon \end{aligned}$$

  or

  $$\begin{aligned} &S_3 \to AW \\ &A \to aA \mid \epsilon \\ &W \to aWc \mid bW \mid \epsilon \end{aligned}$$

- $$S_4 \to aS_4bbb \mid \epsilon$$
- $$T_1 \to S_1S_2$$
- $$T_2 \to S_1 \mid S_2$$

Exercise 2: PDAs *(6 Points)*

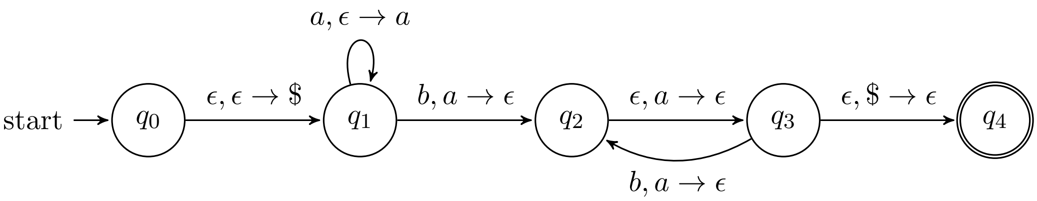
Construct a PDA for the following languages

- L_2 in exercise 1.
- L_3 in exercise 1.
- $L_5 = \{a^n b^{2m} b a^n \mid m, n > 0\}$ over the alphabet $\Sigma = \{a, b\}$.

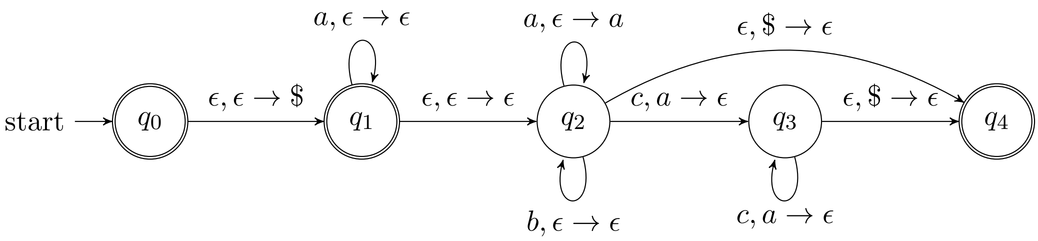
Sample Solution

The formal definition of the automatons is implicitly given.

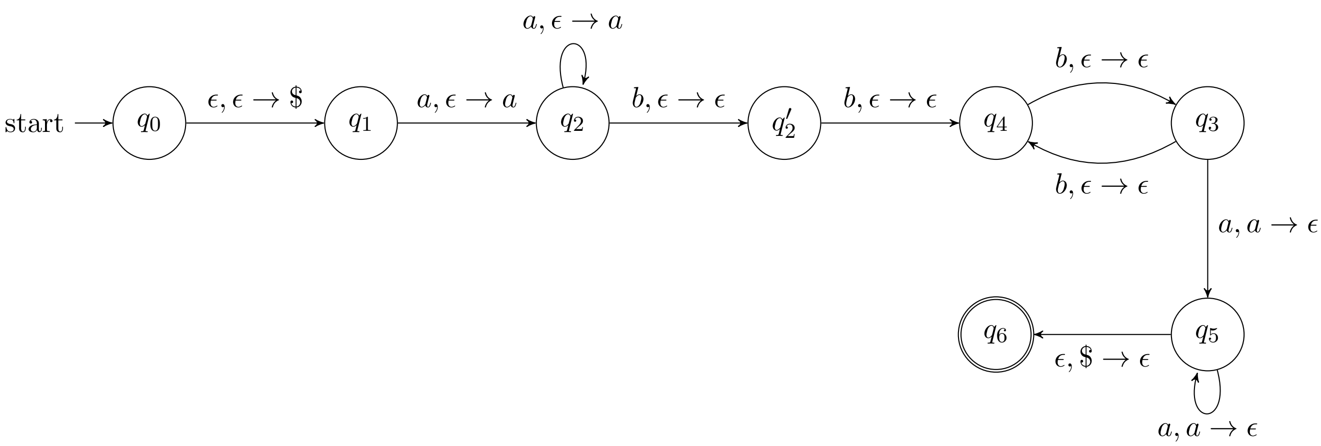
- PDA for L_2:



- PDA for L_3:



- PDA for L_5:



Exercise 3: Chomsky Normal Form *(4 Points)*

Given the following CFG:

$$
\begin{aligned}
S &\to ASA \mid A \mid \epsilon \\
A &\to 00 \mid \epsilon
\end{aligned}
$$

1. What is the language the above CFG recognizes?
2. Convert the following CFG into an equivalent CFG in Chomsky normal form, by following the procedure given in the lecture.

Sample Solution

1. The language is the string of zeros where the number of zero's is even.
2. Applying the procedure from the lecture we get the following equivalent grammar in Chomsky normal form.

$$
\begin{aligned}
S_0 &\to AU_1 \mid SA \mid AS \mid AA \mid UU \mid \epsilon \\
S &\to AU_1 \mid SA \mid AS \mid AA \mid UU \\
A &\to UU \\
U &\to 0 \\
U_1 &\to SA
\end{aligned}
$$

Exercise 4: Closure in CFL *(Bonus Points)*

1. Show that the context-free languages are closed under union, concatenation and Kleene star. *Hint: try to prove that the context-free languages are closed under the above operators via creating the appropriate grammars.*
2. Knowing that $L_6 = \{a^i b^j c^k \mid i < j\}$ is a context free language, are context free languages closed under intersection?
   Hint: Use the fact that $L_7 = \{a^i b^j c^k \mid i < j \text{ and } i < k\}$ is not a context free language.

Sample Solution

1. The context-free languages are closed under union, concatenation and Kleene star, i.e. if L_1 and L_2 are context-free languages, so are $L_1 \cup L_2$, $L_1 L_2$ and L_1^*. Indeed, we will prove that the languages are closed by creating the appropriate grammars. Suppose we have two context-free languages L_1 and L_2 , represented by grammars with start symbols S_1 and S_2 respectively. First of all, rename all the terminal symbols in the second grammar so that they don't conflict with those in the first. Then:

   To get the union, add the rule $S \to S_1 \mid S_2$, with S representing the start symbol of the grammar of $L_1 \cup L_2$.
   To get the concatenation, add the rule $S \to S_1\, S_2$, with S representing the start symbol of the grammar of $L_1 L_2$.
   To get the Kleene star of L_1, add the rule $S \to S_1\, S \mid \varepsilon$ to the grammar for L_1, with S representing the start symbol of the grammar of L_1^*.

2. The context-free languages are not closed under intersection, i.e. if L_1 and L_2 are context-free languages, it it not always true that $L_1 \cap L_2$ is also. We will prove the non-closure of intersection by exhibiting a counter-example. Consider the following two context free languages:

$$\begin{array}{rcl} L_6 & = & \{a^i b^j c^k \quad | \quad i < j\} \\ L_8 & = & \{a^i b^j c^k \quad | \quad i < k\} \end{array}$$

The intersection of these languages is $L_6 \cap L_8 = \{a^i b^j c^k \mid i < j \text{ } and \text{ } i < k\} = L_7$ and we are given that this language is not context-free.

Bonus solution: A grammar for L_8 is as follows

$$\begin{aligned} &S \rightarrow aSc \mid A \\ &A \rightarrow BC \\ &B \rightarrow Bb \mid \epsilon \\ &C \rightarrow Cc \mid c \end{aligned}$$