



Theoretical Computer Science - Bridging Course

Sample Solution Exercise Sheet 11

Due: Tuesday, 15th of July 2025, 12:00 pm

Exercise 1: Construct Formulae

(1+1+1 Points)

Let $\mathcal{S} = \langle \{x, y, z\}, \emptyset, \emptyset, \{R\} \rangle$ be a signature. Translate the following sentences of first order formula over \mathcal{S} into idiomatic English i.e. how would that native speaker read the formulas?

Use $R(x, y)$ as statement ' x is a part of y '.

- (a) $\exists x \forall y R(x, y)$
- (b) $\exists y \forall x R(x, y)$
- (c) $\forall x \forall y \exists z (R(x, z) \wedge R(y, z))$

Sample Solution

Note that idiomatic English uses, contains, or denotes expressions that are natural to a native speaker. It does not contain variables. it might be said: The native speaker will then read the formulas as

- (a) Something is a part of everything.
- (b) Something has everything as a part.
- (c) For any two things, there is something of which they are both a part.

Exercise 2: FOL: Is it a model?

(2+3+3 Points)

Consider the following **first order** formulae

$$\begin{aligned}\varphi_1 &:= \forall x R(x, x) \\ \varphi_2 &:= \forall x \forall y R(x, y) \rightarrow (\exists z R(x, z) \wedge R(z, y)) \\ \varphi_3 &:= \exists x \exists y (\neg R(x, y) \wedge \neg R(y, x))\end{aligned}$$

over signature \mathcal{S} where x, y, z are variable symbols and R is a binary predicate. Give an interpretation

- (a) I_1 which is a **model** of $\varphi_1 \wedge \varphi_2$.
- (b) I_2 which is **no model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.
- (c) I_3 which is a **model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

Sample Solution

- (a) Pick $I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$ where $R^{I_1}(x, y) :\iff x \leq_{\mathbb{R}} y$.

This is a model because ' $\leq_{\mathbb{R}}$ ' is *reflexive*, therefore fulfills φ_1 . Moreover for every $x, y \in \mathbb{R}$ with $x \leq_{\mathbb{R}} y$ we can choose $z := x$, which fulfills $x \leq_{\mathbb{R}} z \wedge z \leq_{\mathbb{R}} y$. Thus φ_2 is also satisfied.

- (b) Let $U = \{1, 2, 3, 4, 5\}$ and $P(U)$ be its power set. Pick $I_2 := \langle P(U), \cdot^I \rangle$ where $R^{I_2}(x, y) :\iff x \subset y$. This is not a model since it doesn't satisfy φ_1 , indeed no set is proper subset of itself.

- (c) Take two disjoint copies of \mathbb{R} and the standard $\leq_{\mathbb{R}}$ relation on each of them; if x and y are from different copies they are not related in \mathbb{R} . Formally let

$$I_3 := \langle \{(a, 1) \mid a \in \mathbb{R}\} \dot{\cup} \{(a, 2) \mid a \in \mathbb{R}\}, \cdot^{I_3} \rangle$$

where $R^{I_3}((a, g), (b, h)) \iff (g = h \text{ and } a \leq_{\mathbb{R}} b)$.

This is a model because $\leq_{\mathbb{R}}$ is *reflexive*, therefore I_3 fulfills φ_1 . Furthermore for every two $x = (a, g)$ and $y = (b, h)$ with $R^{I_3}((a, g), (b, h))$, i.e., $g = h$, we can choose $z := (a, g)$ which fulfills $R^{I_3}((a, g), (a, g)) \wedge R^{I_3}((a, g), (b, h))$. Thus φ_2 is also satisfied. φ_3 is also satisfied, e.g., $(5, 1)$ and $(7, 2)$ are incomparable, i.e., we have neither $R^{I_3}((5, 1), (7, 2))$ nor $R^{I_3}((7, 2), (5, 1))$.

Exercise 3: FOL: Entailment

(3+3+3 Points)

Let φ, ψ be first order formulae over signature \mathcal{S} . Similar to propositional logic, in predicate logic we write $\varphi \models \psi$ if every model of φ is also a model for ψ . We write $\varphi \equiv \psi$ if both $\varphi \models \psi$ and $\psi \models \varphi$. A *knowledge base* KB is a set of formulae. A model of KB is model for all formulae in KB . We write $KB \models \varphi$ if all models of KB are models of φ . Show or disprove the following entailments.

- (a) $(\exists x R(x)) \wedge (\exists x P(x)) \wedge (\exists x T(x)) \models \exists x (R(x) \wedge P(x) \wedge T(x))$.
- (b) $(\forall x \forall y f(x, y) \doteq f(y, x)) \wedge (\forall x f(x, \mathbf{c}) \doteq x) \models \forall x f(\mathbf{c}, x) \doteq x$.
- (c) $(\forall x R(x, x)) \wedge (\forall x \forall y R(x, y) \wedge R(y, x) \rightarrow x \doteq y) \wedge (\forall x \forall y \forall z R(x, y) \wedge R(y, z) \rightarrow R(x, z)) \models \forall x \forall y R(x, y) \vee R(y, x)$.

Hint: Consider order relations. E.g., $a \leq b$ (a less-equal b) and $a \mid b$ (a divides b).

Sample Solution

- (a) The stated entailment is false (it holds in the other direction though). In order to disprove it, we give a model for the left side which is not a model for the right side.

Let $I = \langle \{a, b\}, \cdot^I \rangle$ with $R^I = \{a\}$, $P^I = \{b\}$, and $T^I = \{a\}$. This makes the left side true since there exists an element $x = a$ that makes $R(x)$ and $T(x)$ true and an element $x = b$ that makes $P(x)$ true (Note the brackets around the three \exists quantifiers which mean that the three elements need not necessarily be the same).

However $R(a) \wedge P(a) \wedge T(a) = T \wedge F \wedge T = F$ and $R(b) \wedge P(b) \wedge T(b) = F \wedge T \wedge F = F$ thus the right side is false (there exists no element which makes the three relations' symbols R, P, T true, since we tested all that are in the domain).

- (b) The stated entailment holds. We prove this by picking an arbitrary *model* (!) $I = \langle \mathcal{D}, \cdot^I \rangle$ of the left-hand formula. We show that I is a model for the right-hand formula, too. For that purpose let x be an arbitrary element from \mathcal{D} .

Since I is a model for the left side we already know $f(x, \mathbf{c}^I) \doteq x$. The first condition in the left formula encodes the commutative property. Since \mathbf{c}^I is also an element from the domain \mathcal{D} we know $f(x, \mathbf{c}^I) = f(\mathbf{c}^I, x)$ and thus $f(\mathbf{c}^I, x) \doteq x$. Since x was arbitrary we have $\forall x f(\mathbf{c}^I, x) \doteq x$.

- (c) The formula on the left side encodes the properties of an *order relation*. The formula on the right side encodes the property of *totality* of an order, which means that every element is related to (read: can be compared with) every other element. However, in general an order relation does not need to be total (which is called a *partial order*).

The hint proposes two order relations, one of which is total over the domain of integers \mathbb{Z}^* (either ' $x \leq y$ ' or ' $x \geq y$ ' or both) whereas the other is not (it may happen that neither $x|y$ nor $y|x$). Thus the logical entailment is false since with \mathbb{Z}^* and the 'divides'-relation we have a model of the left-hand formula which is no model of the right-hand one (it is not total).

We formalize this as follows. Let $I = \langle \mathbb{Z}^*, \cdot^I \rangle$ with $R^I := \{(x, y) \in \mathbb{Z}^* \mid x \text{ divides } y\}$. Obviously we have the reflexive property since $x \in \mathbb{Z}^*$ divides itself. If $x \in \mathbb{Z}^*$ divides $y \in \mathbb{Z}^*$ and $y \in \mathbb{Z}^*$ divides $z \in \mathbb{Z}^*$ then x also divides z which gives us transitivity. Finally, if x divides y and vice versa then y is multiple of x and vice versa which means that the multiplicand must in both cases be 1, thus both x and y are equal which gives us the antisymmetry property.

This means that I is a model of the left-hand formula. Now consider the two primes $x = 2$ and $y = 3$. By definition of prime numbers neither of the two can divide the other. Thus $\forall x \forall y R(x, y) \vee R(y, x)$ is false. Therefore I can be no model of the right-hand formula.