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Theoretical Computer Science - Bridging Course
Sample Solution Exercise Sheet 1

Due: Tuesday, 28th of April 2026, 12:00 pm

Exercise 1: Miscellaneous Mathematical Proofs *(10 Points)*

(a) Let $S(n) = \sum_{i=1}^{n} i$ be the sum of the first n natural numbers and $C(n) = \sum_{i=1}^{n} i^3$ be the sum of the first n cubes. Use mathematical induction to prove the following interesting conclusion: $C(n) = S^2(n)$ for every integer $n \geq 0$. *(3 Points)*

(b) Let A, B, and C be subsets of some nonempty universal set U. Which of the following statements is true? Justify.

- If $A \cap B = A \cap C$, then $B = C$. *(2 Points)*
- If $A \cup B = A \cup C$, then $B = C$. *(2 Points)*
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$, where $\overline{A}$ is the compliment of A. *(3 Points)*

Sample Solution

(a) We argue by induction on $n \geq 0$.

Base step: for $n = 1$, $1^3 = (1)^2$ is true.

Induction step: for each $k \geq 1$, we assume that the statement holds true for k i.e. $C(k) = S^2(k)$ (induction hypothesis IH). Now, we need to prove that the statement holds true for $k+1$ i.e. we want to show that $C(k+1) = S^2(k+1)$.

Indeed first, we recall that $S(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, hence $S^2(k+1) = \left(\frac{(k+1)(k+2)}{2}\right)^2 = \frac{(k+1)^2(k+2)^2}{4}$.
Next, we have that $C(k+1) = \sum_{i=1}^{k} i^3 + (k+1)^3 = C(k) + (k+1)^3 \overset{\text{IH}}{=} S^2(k) + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \left(\frac{k^2(k+1)^2}{4}\right) + (k+1)^3 = \frac{(k+1)^2}{4}(k^2+4k+4) = \frac{(k+1)^2}{4}(k+2)^2 = S^2(k+1)$.
Hence, the statement holds true for $k+1$, which ends our induction proof.

(b)

- False. We give a counterexample: take $A = \{1,2,3\}$, $B = \{1,4\}$ and $C = \{1,5\}$, hence $A \cap B = A \cap C$ and $B \neq C$.
- False. We give a counterexample: take $A = \{1,2\}$, $B = \{1,3\}$ and $C = \{2,3\}$, hence $A \cup B = A \cup C$ and $B \neq C$.
- True. Indeed,

  $$x \in \overline{A \cup B} \iff x \notin A \cup B \iff x \notin A \text{ and } x \notin B \iff x \in \overline{A} \text{ and } x \in \overline{B} \iff x \in \overline{A} \cap \overline{B}$$

  hence, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Exercise 2: An Even Degree Node *(4 Points)*

A *simple graph* is a graph without multi-edges (between two nodes there can exist at most one edge) and without self loops (every edge of the graph is an edge between two distinct nodes). Let $G = (V, E)$ be an undirected simple graph. Recall that the degree $d(v)$ of a node $v \in V$ is the number of its neighbors in G , i.e, $d(v) = |\{u \in V \mid \{v,u\} \in E\}|$.

Show that every simple graph with an odd number of nodes contains a node with even degree.

Sample Solution

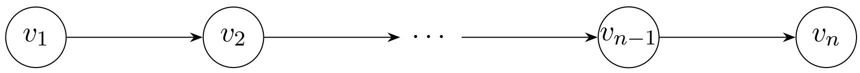
We argue by contradiction. Assume that $\forall v \in V$, $d(v)$ is odd. Now since G has odd number of nodes, we notice that $\sum_{v \in V} d(v)$ is the sum of an odd number of odd numbers, which is odd. But by the handshaking lemma $\sum_{v \in V} d(v)$ must be even. This is a contradiction. Thus our assumption must have been false and hence there must exist a node in G with even degree.

Exercise 3: Visiting All Nodes *(6 Points)*

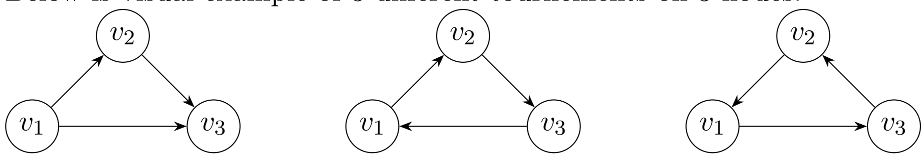
A *complete graph* is a simple undirected graph in which every pair of distinct nodes is connected by a unique edge e.g. a triangle on 3 nodes.

(a) Show that every complete graph G has a path P that visits all the nodes of G. *(1 Point)*

A *directed path* P on n vertices is a simple directed graph whose edge set is the following set of ordered pairs $\{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1 \text{ and } v_i \text{ is a node in } P\}$ i.e. a path in which all the arrows point in the same direction as its steps . We write $P = v_1 v_2 ... v_n$ to denote the directed path P, where below is a visual example.



A *tournament* is a complete graph that is oriented, or equivalently a directed graph in which every pair of distinct vertices is connected by a directed edge with any one of the two possible orientations. Below is visual example of 3 different tournements on 3 nodes.



Tournament 1 Tournament 2 Tournament 3

(b) Prove that every tournament T has a directed path P that visits all the nodes of T. *(5 Points)*

Hint: Prove by contradiction. Consider a longest directed path in T and suppose that this path doesn't visit all nodes in T. What happens then?

Sample Solution

(a) We prove existence of path P by construction as follows. First, pick any node u in G. Since G is a complete graph, there exists an edge from u to each one of the other nodes, different from u. Second, choose one of these nodes say v, hence $\{u, v\}$ is an edge in G and will be the first edge in the construction of P. Then, delete u from G and name the new graph G_1. Notice G_1 is still a complete graph. Pick any node x in G_1 other than v, hence $\{v, x\}$ is an edge in G_1, thus in G, and will be the second edge in the construction of P. Repeat this process until only one node is left. Therefore, path P is constructed and visits all nodes of G.

(b) Let $T = (V, E)$ be a tournament on n nodes and $P = v_1 ... v_s$ be a longest directed path in T, where $v_i \in V$ for $1 \leq i \leq s$.
Assume that $s \neq n$, then there exists a node u in T and not in P. Notice that since T is a complete graph, there is an edge from u to v_i, for all $1 \leq i \leq s$. However, we still don't know whether (u, v_i) or (v_i, u) is in E. Now, if $(u, v_1) \in E$, then $P^{'} = u v_1 v_2 ... v_s$ is a directed path longer than P, a contradiction. So, it is necessarily that $(v_1, u) \in E$. Similarly, we show that $(u, v_s) \in E$ necessarily. Then, there exists $1 \leq i_0 \leq s$ such that (v_{i_0}, u) and (u, v_{i_0+1}) are both in E. Therefore, path

$P^{'} = v_1 ... v_{i_0} u v_{i_0+1} ... v_s$ is a directed path longer than P, a contradiction. Hence, s must only be equal to n i.e. P is a directed path visiting all the nodes of T.