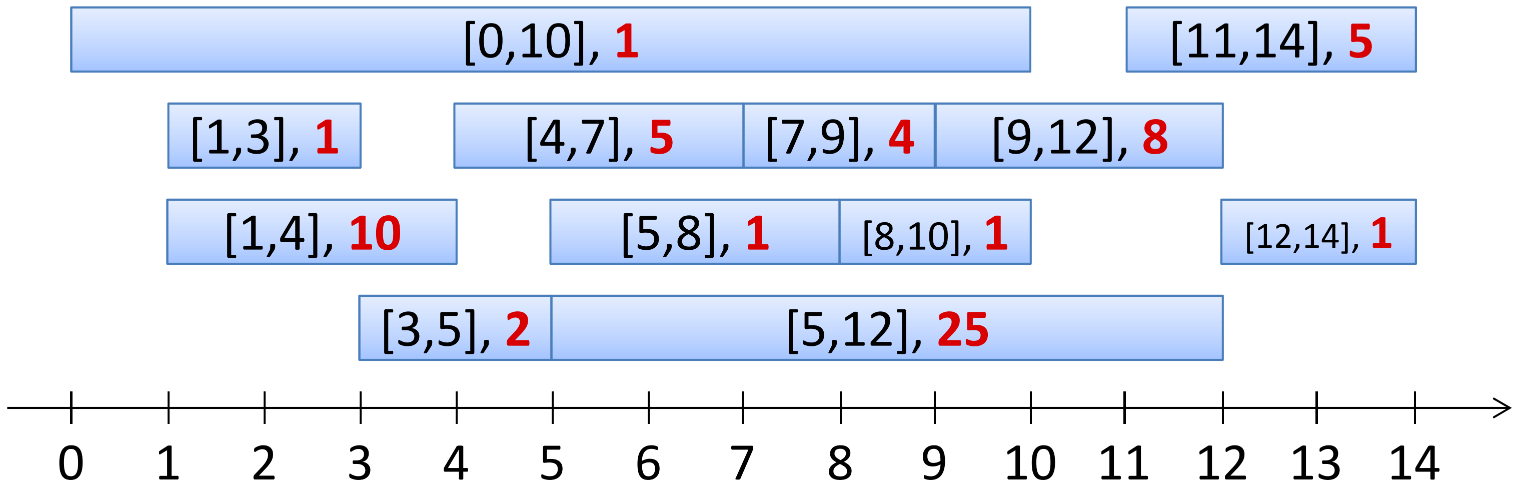
Chapter 3
Dynamic Programming

Algorithm Theory
WS 2013/14

Fabian Kuhn

Weighted Interval Scheduling

- **Given:** Set of intervals, e.g.
  [0,10],[1,3],[1,4],[3,5],[4,7],[5,8],[5,12],[7,9],[9,12],[8,10],[11,14],[12,14]
- Each interval has a **weight *w***

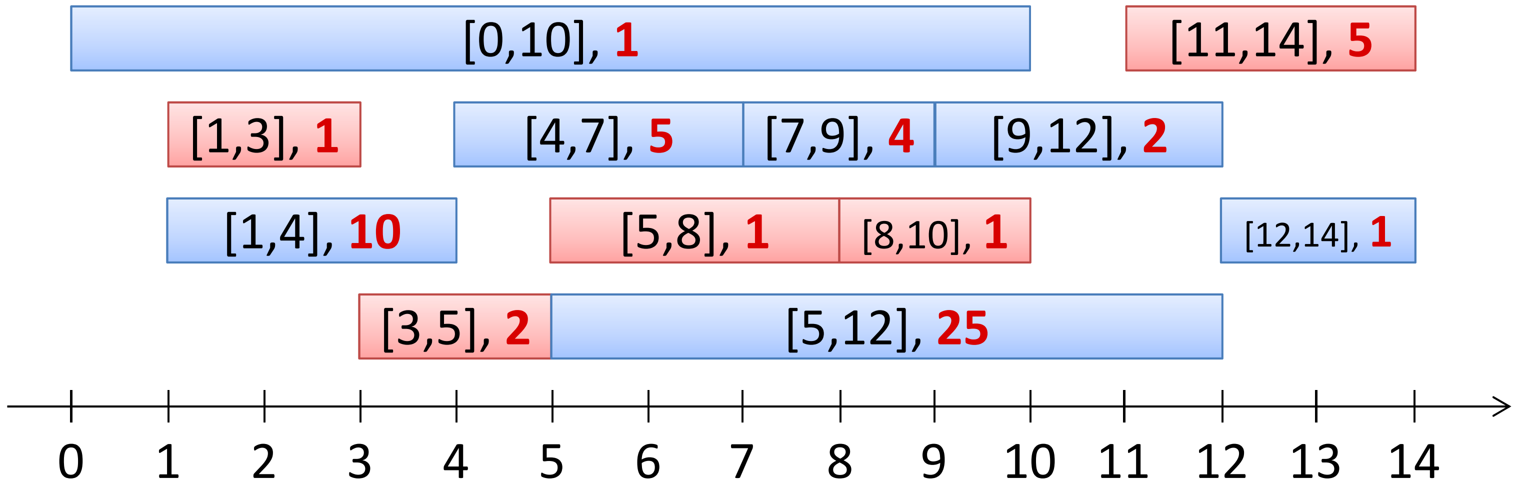


- **Goal:** Non-overlapping set of intervals of largest possible weight
  – Overlap at boundary ok, i.e., [4,7] and [7,9] are non-overlapping

- **Example:** Intervals are room requests of different importance

Greedy Algorithms

Choose available request with earliest finishing time:



- Algorithm is not optimal any more
  - It can even be arbitrarily bad…

- No greedy algorithm known that works

Solving Weighted Interval Scheduling

- Interval i: start time $s(i)$, finishing time: $f(i)$, weight: $w(i)$

- Assume intervals $1, \dots, n$ are sorted by increasing $f(i)$
  - $0 < f(1) \leq f(2) \leq \cdots \leq f(n)$, for convenience: $f(0) = 0$

- Simple observation:
  Opt. solution contains interval n or it doesn't contain interval n

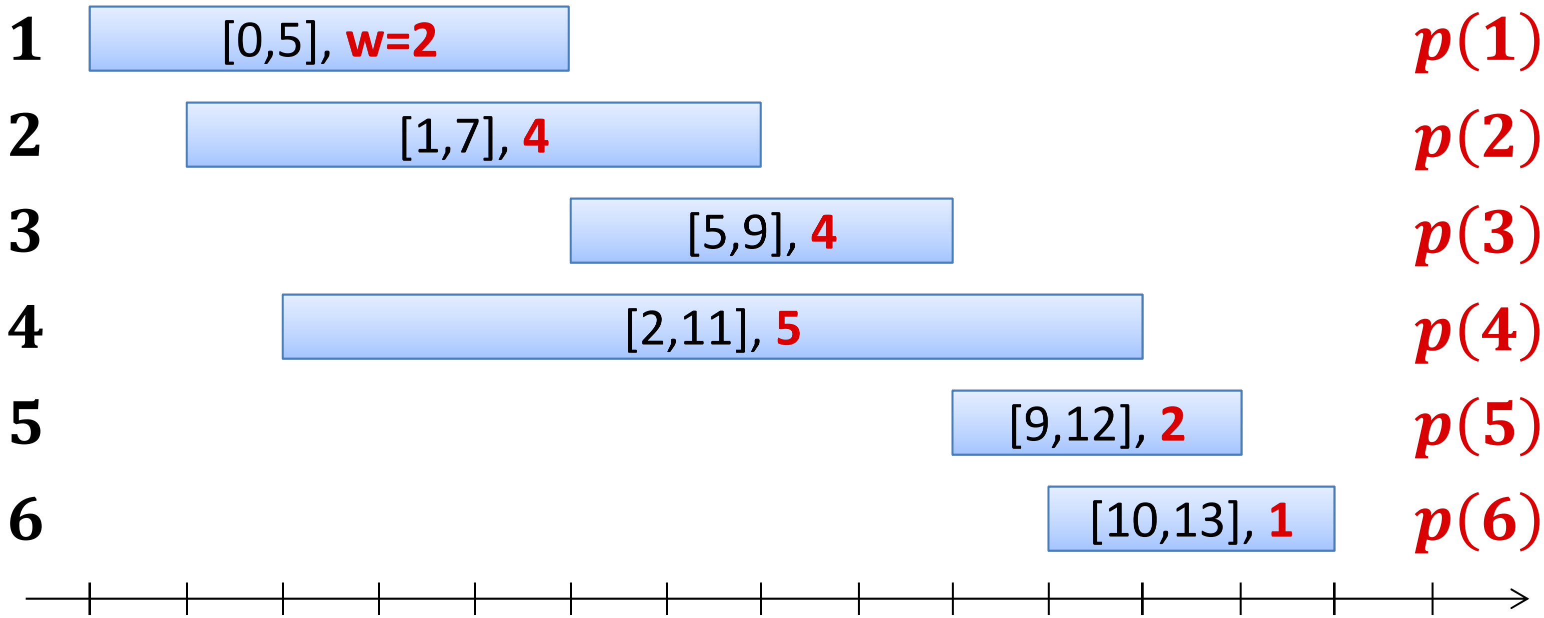
- Weight of optimal solution for only intervals $1, \dots, k$: $W(k)$
  Define $p(k) := \max\{i \in \{0, \dots, k-1\} : f(i) \leq s(k)\}$

- Opt. solution does not contain interval n: $\boldsymbol{W(n) = W(n-1)}$

  Opt. solution contains interval n: $\boldsymbol{W(n) = w(n) + W(p(n))}$

Example

Interval:



$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 0$

$p(5) = 3$

$p(6) = 3$

Recursive Definition of Optimal Solution

- Recall:
  - $W(k)$: weight of optimal solution with intervals $1, \dots, k$
  - $p(k)$: last interval to finish before interval k starts

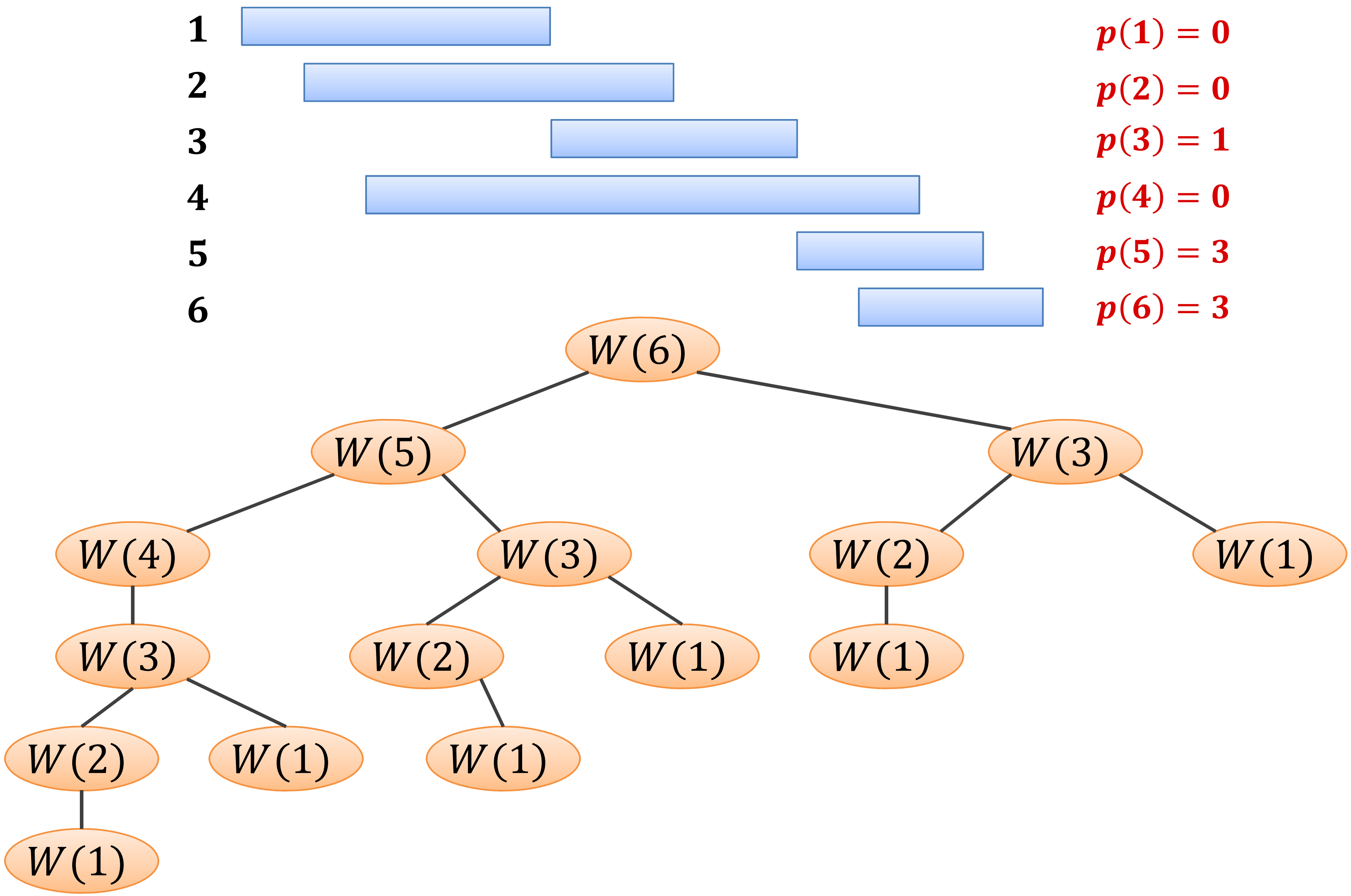
- Recursive definition of optimal weight:

$$\forall k > 1: \; W(k) = \max\{W(k-1), w(k) + W\big(p(k)\big)\}$$

$$W(1) = w(1)$$

- Immediately gives a simple, recursive algorithm

Running Time of Recursive Algorithm



Memoizing the Recursion

- Running time of recursive algorithm: exponential!
- But, alg. only solves n different sub-problems: $W(1), \dots, W(n)$
- There is no need to compute them multiple times

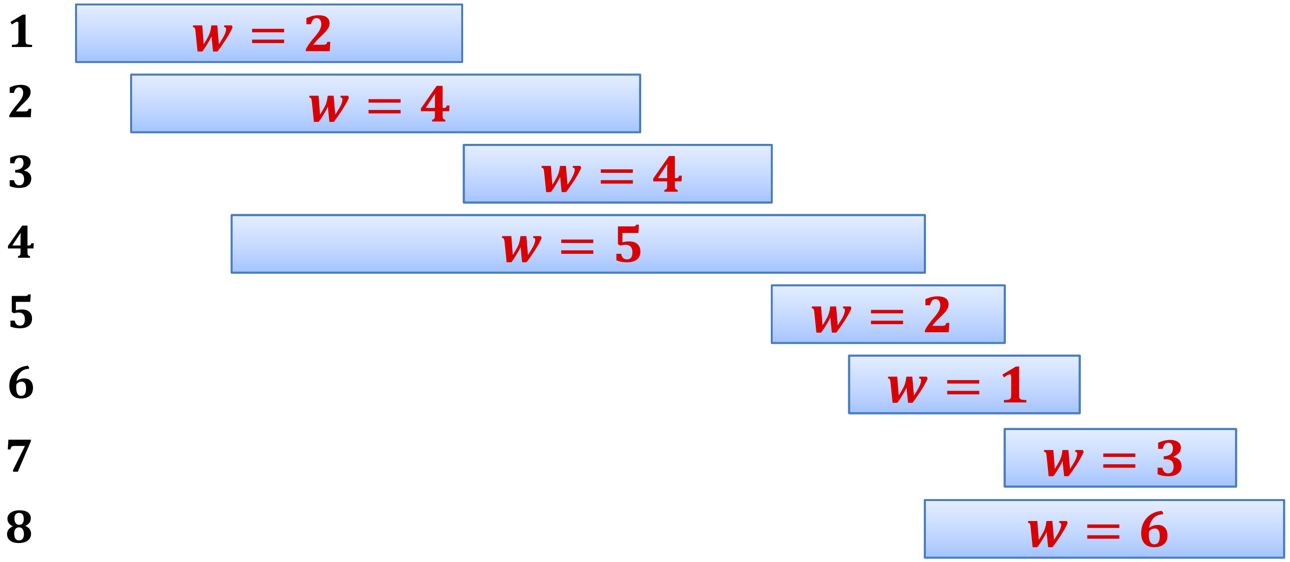
Memoization:

- **Store already computed values** for future use (recursive calls)

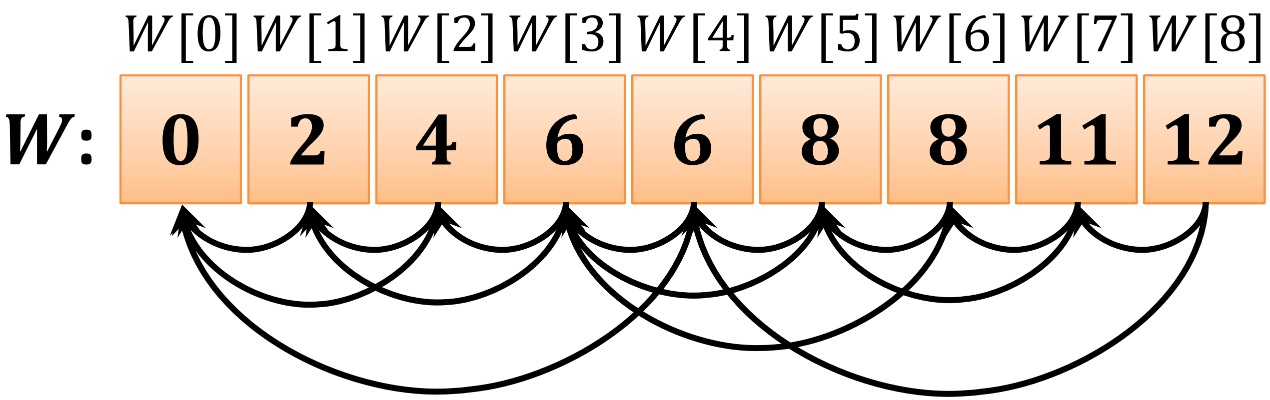
Efficient algorithm:

1. $W[0] \coloneqq 0$; compute values $p(i)$
2. **for** $i \coloneqq 1$ **to** n **do**
3. $\quad W[i] \coloneqq \max\{W[i-1], w(i) + W[p(i)]\}$
4. **end**

Example



$p(1) = 0$
$p(2) = 0$
$p(3) = 1$
$p(4) = 0$
$p(5) = 3$
$p(6) = 3$
$p(7) = 5$
$p(8) = 4$



Computing the schedule: **store where you come from!**

Matrix-chain multiplication

Given: sequence (chain) $\langle A_1, A_2, \ldots, A_n \rangle$ of matrices

Goal: compute the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$

Problem: Parenthesize the product in a way that minimizes the number of scalar multiplications.

Definition: A product of matrices is *fully parenthesized* if it is

- a single matrix
- or the product of two fully parenthesized matrix products, surrounded by parentheses.

Example

All possible fully parenthesized matrix products of the chain $\langle A_1, A_2, A_3, A_4 \rangle$:

$$(A_1(A_2(A_3A_4)))$$

$$(A_1((A_2A_3)A_4))$$

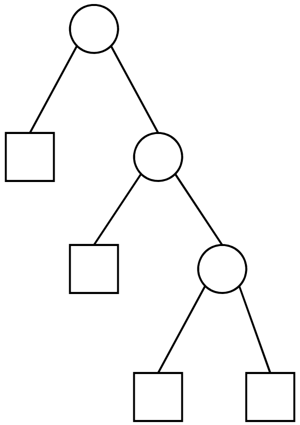
$$((A_1A_2)(A_3A_4))$$

$$((A_1(A_2A_3))A_4)$$

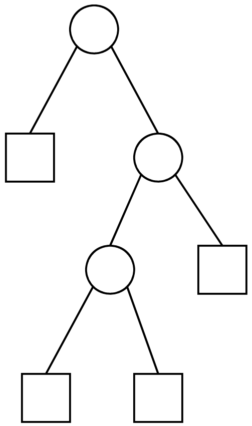
$$(((A_1A_2)A_3)A_4)$$

Different parenthesizations

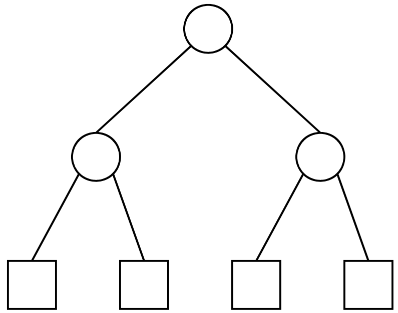
Different parenthesizations correspond to different trees:



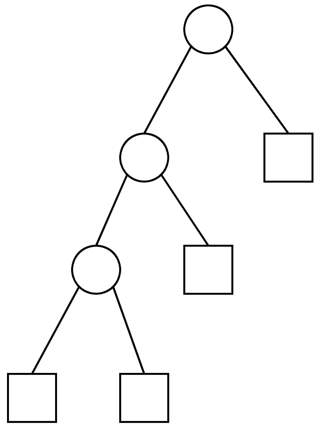
$(A_1(A_2(A_3A_4)))$



$(A_1((A_2A_3)A_4))$



$((A_1A_2)(A_3A_4))$



$(((A_1A_2)A_3)A_4)$

Number of different parenthesizations

- Let $P(n)$ be the number of alternative parenthesizations of the product $A_1 \cdot \ldots \cdot A_n$:

$$P(1) = 1$$

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k), \qquad \text{for } n \geq 2$$

$$P(n+1) = \frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}} + O\left(\frac{4^n}{\sqrt{n^5}}\right)$$

$$P(n+1) = C_n \qquad (n^{th} \text{ Catalan number})$$

- Thus: Exhaustive search needs exponential time!

Multiplying Two Matrices

$$A = \left(a_{ij}\right)_{p\times q}, \qquad B = \left(b_{ij}\right)_{q\times r}, \qquad A \cdot B = C = \left(c_{ij}\right)_{p\times r}$$

$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$$

Algorithm *Matrix-Mult*

Input: $(p \times q)$ matrix A, $(q \times r)$ matrix B

Output: $(p \times r)$ matrix $C = A \cdot B$

```
for i := 1 to p do
    for j := 1 to r do
        C[i,j] := 0;
        for k := 1 to q do
            C[i,j] := C[i,j] + A[i,k] · B[k,j]
```

Remark:

Using this algorithm, multiplying two $(n \times n)$ matrices requires n^3 multiplications. This can also be done using $O(n^{2.376})$ multiplications.

Number of multiplications and additions: $\boldsymbol{p \cdot q \cdot r}$

Matrix-chain multiplication: Example

Computation of the product $A_1 A_2 A_3$, where

A_1 : (50×5) matrix

A_2 : (5×100) matrix

A_3 : (100×10) matrix

a) Parenthesization $((A_1 A_2)A_3)$ and $(A_1(A_2A_3))$ require:

$A' = (A_1 A_2)$:

$A'A_3$:

$A'' = (A_2A_3)$:

A_1A'':

Sum:

Structure of an Optimal Parenthesization

- $(A_{\ell\ldots r})$: optimal parenthesization of $A_\ell \cdot \ldots \cdot A_r$

  For some $1 \leq k < n$: $\boldsymbol{(A_{1\ldots n}) = \big((A_{1\ldots k}) \cdot (A_{k+1\ldots n})\big)}$

- Any optimal solution contains optimal solutions for sub-problems

- Assume matrix A_i is a $(d_{i-1} \times d_i)$-matrix

- Cost to solve sub-problem $A_\ell \cdot \ldots \cdot A_r, \ell \leq r$ optimally: $C(\ell, r)$

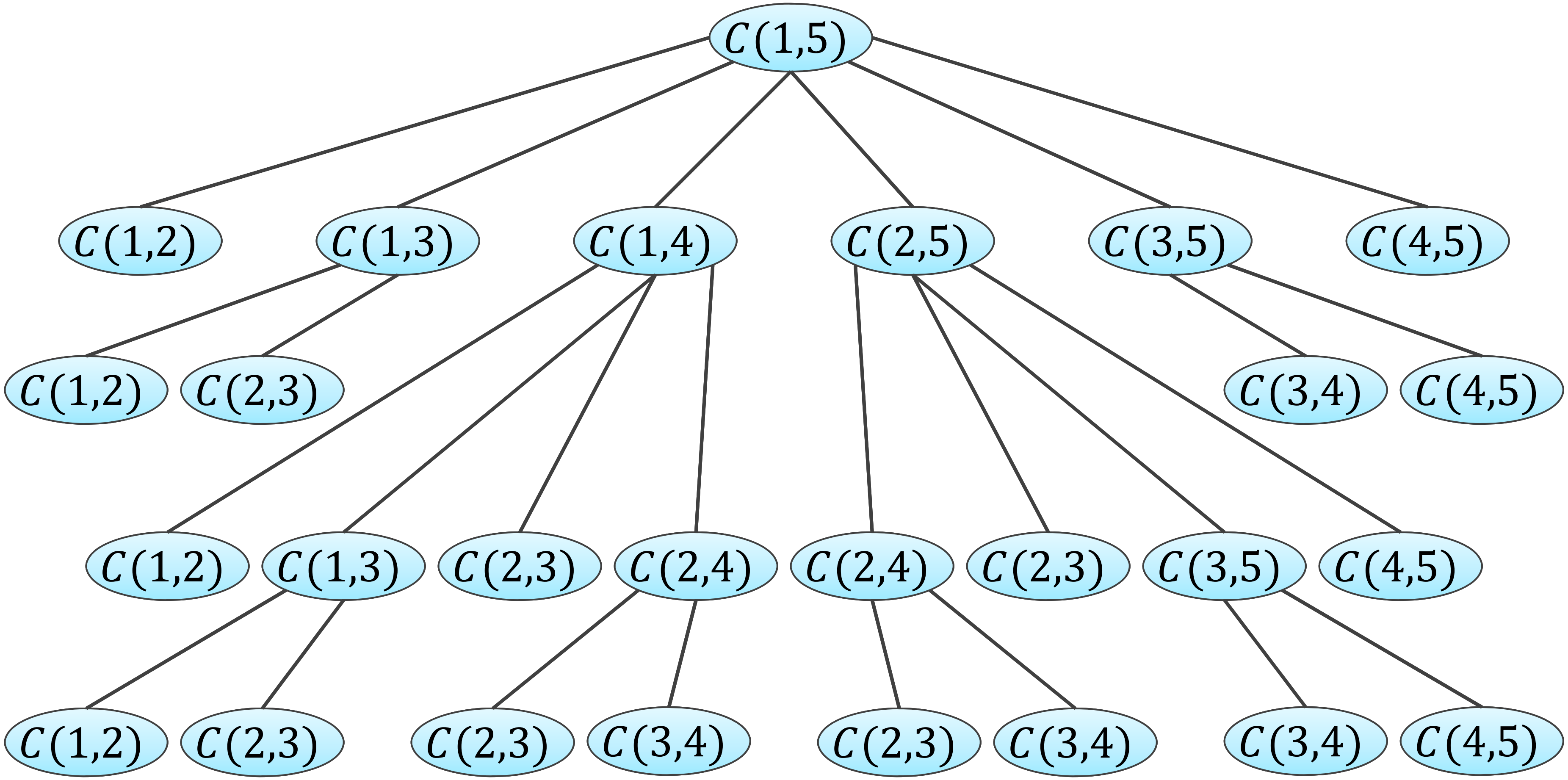
- Then:

$$\boldsymbol{C(a,b) = \min_{a \leq k < b} C(a,k) + C(k+1,b) + d_{a-1} d_k d_b}$$

$$\boldsymbol{C(a,a) = 0}$$

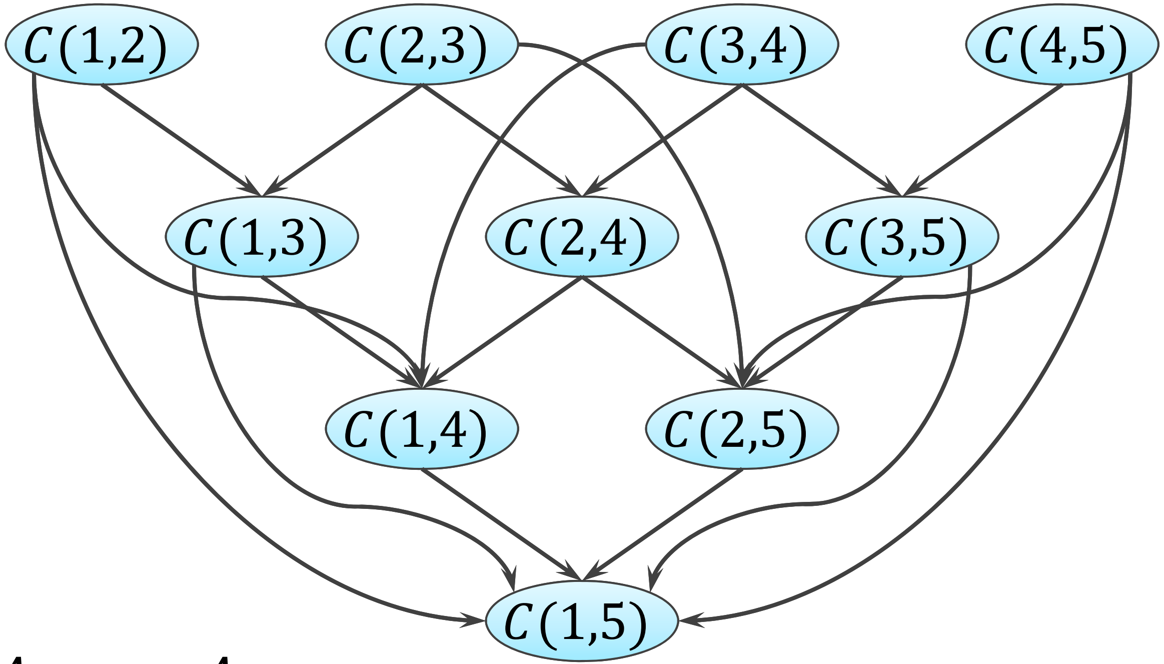
Recursive Computation of Opt. Solution

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



Using Meomization

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



Compute $A_1 \cdot \ldots \cdot A_n$:

- Each $C(i,j), i < j$ is computed exactly once → $O(n^2)$ values
- Each $C(i,j)$ dir. depends on $C(i,k), C(k,j)$ for $i < k < j$

Cost for each $C(i,j)$: $O(n)$ → overall time: $\boldsymbol{O(n^3)}$

Dynamic Programming

„Memoization“ for increasing the efficiency of a recursive solution:

- Only the *first time* a sub-problem is encountered, its solution is computed and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned
  (without repeated computation!).

- *Computing the solution*: For each sub-problem, store how the value is obtained (according to which recursive rule).

Dynamic Programming

Dynamic programming / memoization can be applied if

- Optimal solution contains optimal solutions to sub-problems (recursive structure)

- Number of sub-problems that need to be considered is small

Remarks about matrix-chain multiplication

1. There is an algorithm that determines an optimal parenthesization in time

$$O(n \cdot \log n).$$

2. There is a linear time algorithm that determines a parenthesization using at most

$$1.155 \cdot C(1, n)$$

multiplications.

Knapsack

- n items $1, \dots, n$, each item has weight w_i and value v_i
- Knapsack (bag) of capacity W

- Goal: pack items into knapsack such that total weight is at most W and total value is maximized:

$$\max \sum_{i \in S} v_i$$

$$\text{s.t.} \;\; S \subseteq \{1, \dots, n\} \text{ and } \sum_{i \in S} w_i \leq W$$

- E.g.: jobs of length w_i and value v_i, server available for W time units, try to execute a set of jobs that maximizes the total value

Recursive Structure?

- Optimal solution: $\mathcal{O}$
- If $n \notin \mathcal{O}$: $\mathrm{OPT}(n) = \mathrm{OPT}(n-1)$
- What if $n \in \mathcal{O}$?
  - Taking n gives value v_n
  - But, n also occupies space w_n in the bag (knapsack)
  - There is space for $W - w_n$ total weight left!

$$\mathrm{OPT}(n) = w_n + \begin{array}{l}\text{optimal solution with first } n-1 \text{ items} \\ \text{and knapsack of capacity } W - w_n\end{array}$$

A More Complicated Recursion

$\mathbf{OPT}(\boldsymbol{k}, \boldsymbol{x})$: value of optimal solution with items $1, \dots, k$
and knapsack of capacity x

Recursion:

Dynamic Programming Algorithm

Set up table for all possible $\mathrm{OPT}(k, x)$-values

- Assume that all weights w_i are integers!

	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$			$\boldsymbol{\cdots}$						$\boldsymbol{W}$
$\mathbf{1}$												
$\mathbf{2}$												
$\mathbf{3}$												
$\boldsymbol{\vdots}$												
$\boldsymbol{n}$												

Row $\boldsymbol{i}$, column $\boldsymbol{j}$:

$\boldsymbol{OPT(i, j)}$

Example

- 8 items: (3,2), (2,4), (4,1), (5,6), (3,3), (4,3), (5, 4), (6,6)
  Knapsack capacity: 12
  weight value
- $OPT(k,x) = \max\{OPT(k-1,x), OPT(k-1,x-w_k)+v_k\}$

	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4												
5												
6												
7												
8												

Running Time of Knapsack Algorithm

- **Size of table:** $O(n \cdot W)$

- Time per table entry: $O(1)$ → **overall time:** $\boldsymbol{O(nW)}$

- Computing solution (set of items to pick):
  Follow $\leq n$ arrows → $O(n)$ time (after filling table)

- Note: Time depends on W → can be exponential in n…
- And it is problematic if weights are not integers.

String Matching Problems

Edit distance:

- For two given strings A and B, efficiently compute the

  edit distance $D(A, B)$ (# edit operations to transform A into B)

  as well as a minimum sequence of edit operations that transform A into B.

- **Example:** mathematician → multiplication:

m u t i p l a t i o ~~i~~ ~~a~~ n

l i c

String Matching Problems

Edit distance $\boldsymbol{D(A,B)}$ (between strings A and B):

```
m a – t h e m - - a t i c i a n
m u l t i p l i c a t i o - - n
```

Approximate string matching:

For a given text *T*, a pattern *P* and a distance *d*, find all

substrings P' of T with $D(P,P') \leq d$.

Sequence alignment:

Find optimal alignments of DNA / RNA / ... sequences.

```
G A G C A - C T T G G A T T C T C G G
- - - C A C G T G G - A - A C T - - -
```

Edit Distance

Given: Two strings $A = a_1 a_2 \dots a_m$ and $B = b_1 b_2 \dots b_n$

Goal: Determine the minimum number $D(A, B)$ of edit operations required to transform A into B

Edit operations:

a) **Replace** a character from string A by a character from B

b) **Delete** a character from string A

c) **Insert** a character from string B into A

```
m a – t h e m - - a t i c i a n
m u l t i p l i c a t i o - - n
```

Edit Distance – Cost Model

- Cost for **replacing** character a by b: $\boldsymbol{c(a,b) \geq 0}$

- Capture insert, delete by allowing $a = \varepsilon$ or $b = \varepsilon$:
  – Cost for **deleting** character a: $\boldsymbol{c(a,\varepsilon)}$
  – Cost for **inserting** character b: $\boldsymbol{c(\varepsilon,b)}$

- **Triangle inequality**:

$$c(a,c) \leq c(a,b) + c(b,c)$$

  → each character is changed at most once!

- **Unit cost model**: $c(a,b) = \begin{cases} 1, & \text{if } a \neq b \\ 0, & \text{if } a = b \end{cases}$

Recursive Structure

- Optimal "alignment" of strings (unit cost model)

  `bbcadfagikccm` and `abbagflrgikacc`:

```
- b b c a g f a – g i k - c c m
a b b – a d f l r g i k a c c –
```

- Consists of optimal "alignments" of sub-strings, e.g.:

```
-bbcagfa         –gik-ccm
abb-adfl   and   rgikacc-
```

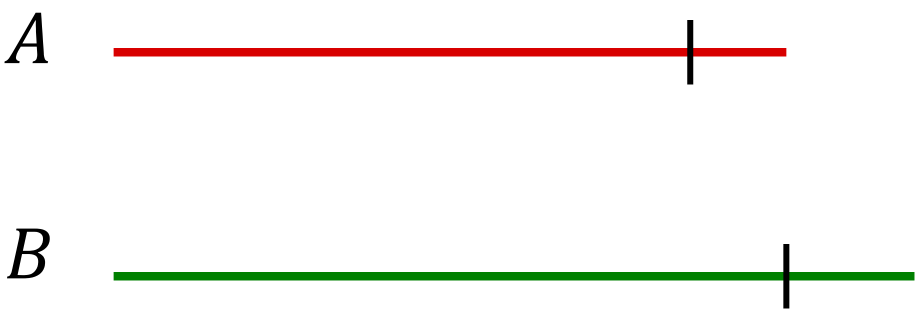
- Edit distance between $A_{1,m} = a_1 \dots a_m$ and $B_{1,n} = b_1 \dots b_n$:

$$D(A,B) = \min_{k,\ell}\left\{D\left(A_{1,k}, B_{1,\ell}\right) + D\left(A_{k+1,m}, B_{\ell+1,n}\right)\right\}$$

Computation of the Edit Distance

Let $A_k := a_1 \dots a_k$, $B_\ell := b_1 \dots b_\ell$, and

$$D_{k,\ell} := D(A_k, B_\ell)$$



Computation of the Edit Distance

Three ways of ending an "alignment" between A_k and B_ℓ:

1. a_k is replaced by b_ℓ:

   $$D_{k,\ell} = D_{k-1,\ell-1} + c(a_k, b_\ell)$$

2. a_k is deleted:

   $$D_{k,\ell} = D_{k-1,\ell} + c(a_k, \varepsilon)$$

3. b_ℓ is inserted:

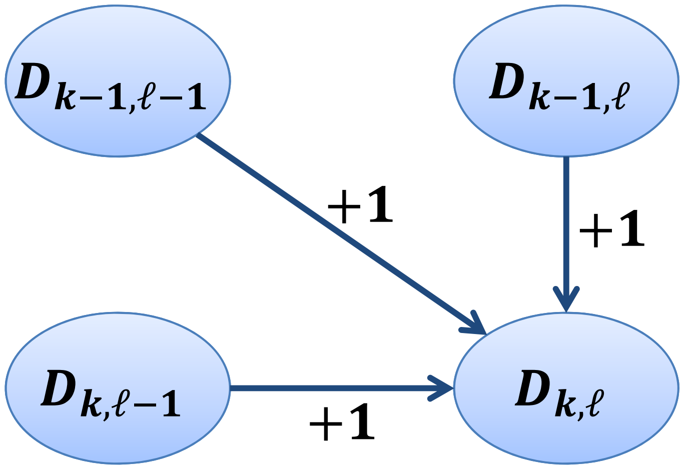
   $$D_{k,\ell} = D_{k,\ell-1} + c(\varepsilon, b_\ell)$$

Computing the Edit Distance

- Recurrence relation (for $k, \ell \geq 1$)

$$D_{k,\ell} = \min \begin{Bmatrix} D_{k-1,\ell-1} & + c(a_k, b_\ell) \\ D_{k-1,\ell} & + c(a_k, \varepsilon) \\ D_{k,\ell-1} & + c(\varepsilon, b_\ell) \end{Bmatrix} = \underbrace{\min \begin{Bmatrix} D_{k-1,\ell-1} & + 1 \,/\, 0 \\ D_{k-1,\ell} & + 1 \\ D_{k,\ell-1} & + 1 \end{Bmatrix}}_{\textbf{unit cost model}}$$

- Need to compute $D_{i,j}$ for all $0 \leq i \leq k$, $0 \leq j \leq \ell$:



Recurrence Relation for the Edit Distance

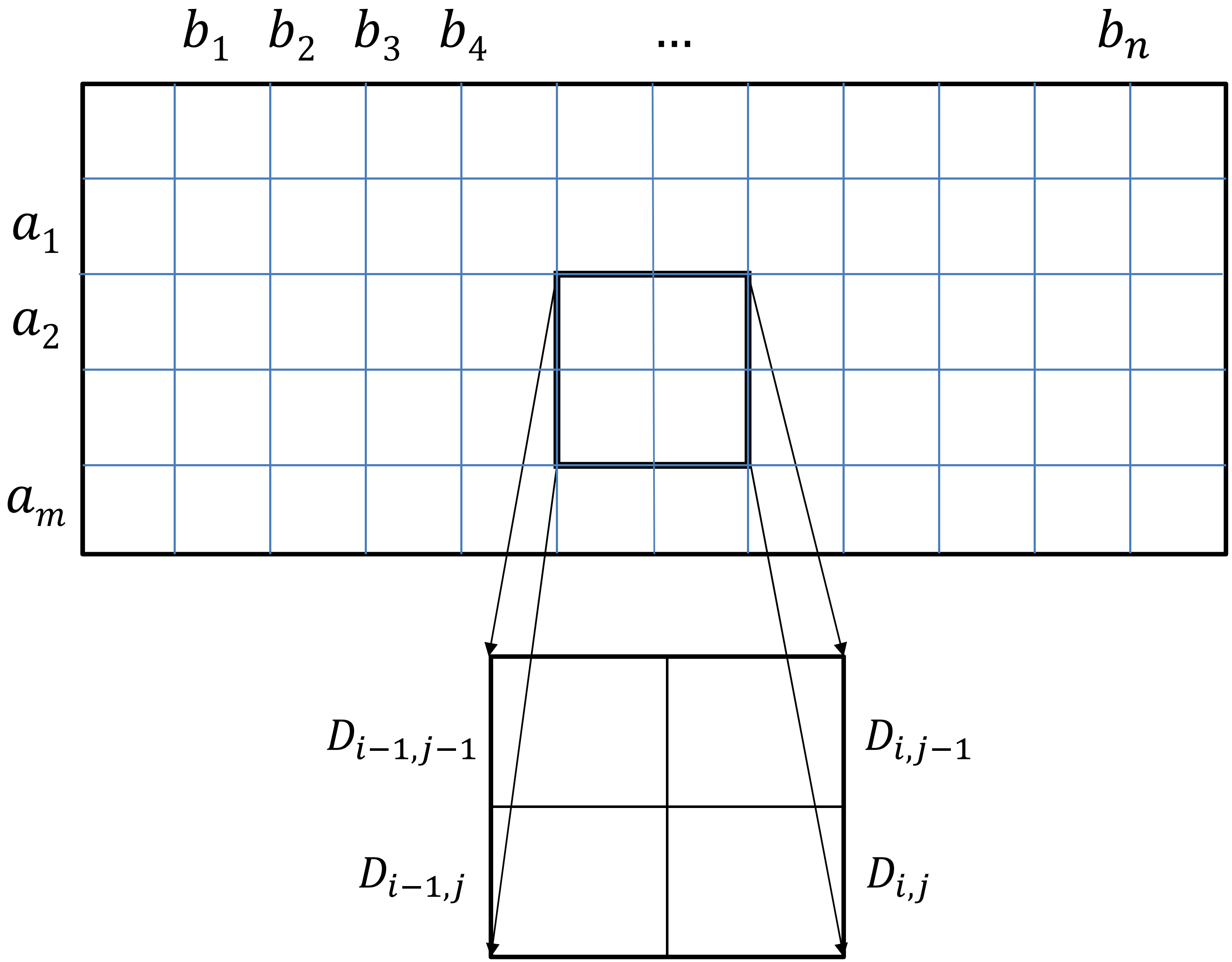
Base cases:

$$D_{0,0} = D(\varepsilon, \varepsilon) = 0$$
$$D_{0,j} = D(\varepsilon, B_j) = D_{0,j-1} + c(\varepsilon, b_j)$$
$$D_{i,0} = D(A_i, \varepsilon) = D_{i-1,0} + c(a_i, \varepsilon)$$

Recurrence relation:

$$D_{i,j} = \min \begin{Bmatrix} D_{k-1,\ell-1} & + c(a_k, b_\ell) \\ D_{k-1,\ell} & + c(a_k, \varepsilon) \\ D_{k,\ell-1} & + c(\varepsilon, b_\ell) \end{Bmatrix}$$

Order of solving the subproblems



Algorithm for Computing the Edit Distance

Algorithm *Edit-Distance*

Input: 2 strings $A = a_1 \dots a_m$ and $B = b_1 \dots b_n$

Output: matrix $D = \left(D_{ij}\right)$

$D[0,0] \coloneqq 0;$

for $i \coloneqq 1$ **to** m **do** $D[i,0] \coloneqq i;$

for $j \coloneqq 1$ **to** n **do** $D[0,j] \coloneqq j;$

for $i \coloneqq 1$ **to** m **do**

  for $j \coloneqq 1$ **to** n **do**

$$D[i,j] \coloneqq \min \left\{ \begin{array}{ll} D[i-1,j] & +1 \\ D[i,j-1] & +1 \\ D[i-1,j-1] & + c\left(a_i, b_j\right) \end{array} \right\};$$

Example

		a	b	c	c	a
b						
a						
b						
d						
a						

Computing the Edit Operations

Algorithm *Edit-Operations*(i, j)
Input: matrix D (already computed)
Output: list of edit operations

if $i = 0$ **and** $j = 0$ **then return** empty list

if $i \neq 0$ **and** $D[i, j] = D[i-1, j] + 1$ **then**
    return *Edit-Operations*$(i-1, j) \circ$ „delete a_i“

else if $j \neq 0$ **and** $D[i, j] = D[i, j-1] + 1$ **then**
    return *Edit-Operations*$(i, j-1) \circ$ „insert b_j“

else // $D[i, j] = D[i-1, j-1] + c(a_i, b_j)$
    if $a_i = b_i$ **then return** *Edit-Operations*$(i-1, j-1)$
    else return *Edit-Operations*$(i-1, j-1) \circ$ „replace a_i by b_j“

Initial call: *Edit-Operations*(*m*,*n*)

Edit Operations

		a	b	c	c	a
	0	1	2	3	4	5
b	1	1	1	2	3	4
a	2	1	2	2	3	3
b	3	2	1	2	3	4
d	4	3	2	2	3	4
a	5	4	3	3	3	3

Edit Distance: Summary

- Edit distance between two strings of length m and n can be computed in $O(mn)$ time.

- Obtain the edit operations:
  - for each cell, store which rule(s) apply to fill the cell
  - track path backwards from cell (m, n)
  - can also be used to get all optimal "alignments"

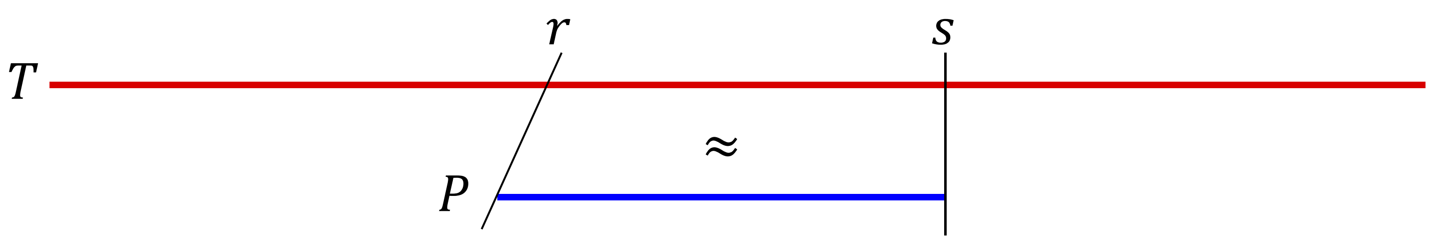
- Unit cost model:
  - interesting special case
  - each edit operation costs 1

Approximate String Matching

Given: strings $T = t_1 t_2 \dots t_n$ (text) and $P = p_1 p_2 \dots p_m$ (pattern).

Goal: Find an interval $[r, s]$, $1 \leq r \leq s \leq n$ such that the sub-string $T_{r,s} := t_r \dots t_s$ is the one with highest similarity to the pattern P:

$$\arg\min_{1 \leq r \leq s \leq n} D\left(T_{r,s}, P\right)$$



Approximate String Matching

Naive Solution:

for all $1 \leq r \leq s \leq n$ **do**

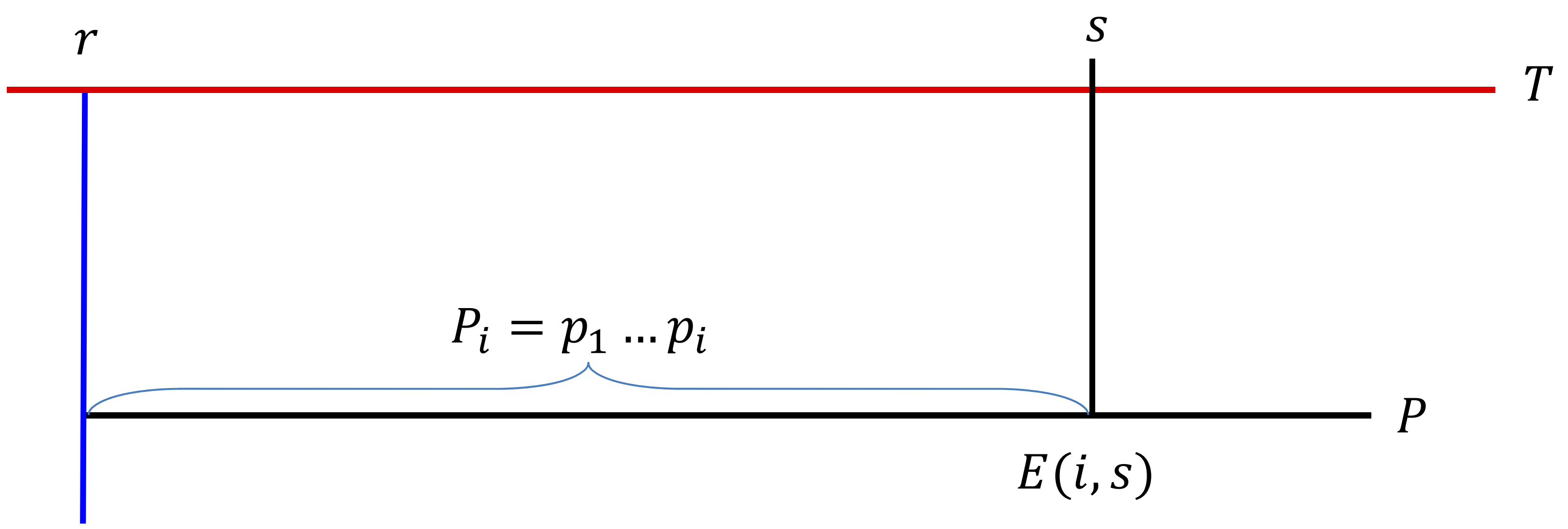
compute $D(T_{r,s}, P)$

choose the minimum

Approximate String Matching

A related problem:

- For each position s in the text and each position i in the pattern compute the minimum edit distance $E(i,s)$ between $P_i = p_1 \dots p_i$ and any substring $T_{r,s}$ of T that ends at position s.



Approximate String Matching

Three ways of ending optimal alignment between T_b and P_i:

1. t_b is replaced by p_i:

   $E_{b,i} = E_{b-1,i-1} + c(t_b, p_i)$

2. t_b is deleted:

   $E_{b,i} = E_{b-1,i} + c(t_b, \varepsilon)$

3. p_i is inserted:

   $E_{b,i} = E_{b,i-1} + c(\varepsilon, p_i\ )$

Approximate String Matching

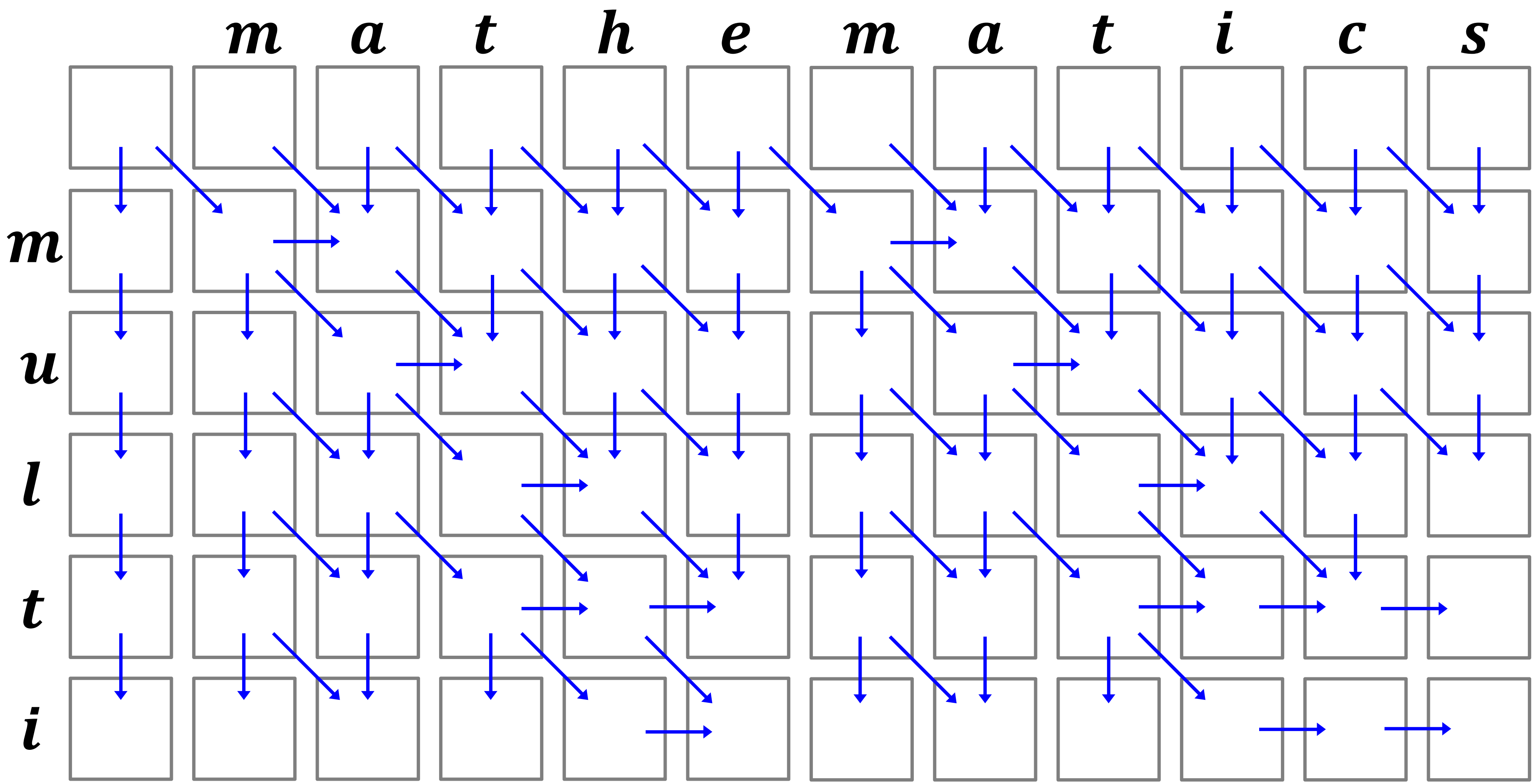
Recurrence relation (unit cost model):

$$E_{b,i} = \min \left\{ \begin{matrix} E_{b-1,i-1} + 1 \\ E_{b-1,i} \quad + 1 \\ E_{b,i-1} \quad + 1 \end{matrix} \right\}$$

Base cases:

$$E_{0,0} = 0$$
$$E_{0,i} = i$$
$$E_{i,0} = 0$$

Example



Approximate String Matching

- Optimal matching consists of optimal sub-matchings

- Optimal matching can be computed in $O(mn)$ time

- Get matching(s):
  - Start from minimum entry/entries in bottom row
  - Follow path(s) to top row

- Algorithm to compute $E(b, i)$ identical to edit distance algorithm, except for the initialization of $E(b, 0)$

Related Problems from Bioinformatics

Sequence Alignment:

Find optimal alignment of two given DNA, RNA, or amino acid sequences.

```
G A – C G G A T T A G
G A T C G G A A T - G
```

Global vs. Local Alignment:

- *Global alignment*: find optimal alignment of 2 sequences
- *Local alignment*: find optimal alignment of sequence 1 (patter) with sub-sequence of sequence 2 (text)