Albert-Ludwigs-Universität, Inst. für Informatik
Prof. Dr. Fabian Kuhn
H. Ghodselahi, Y. Maus
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Algorithm Theory, Winter Term 2014/15
Problem Set 9

hand in (hard copied) by Thursday, 10:00, January 08, 2015, either before the lecture or in the box corresponding to your group in building no. 51.

Exercise 0: Max Flow (Short) Questions (0+0+0+1 points)

Questions a)-c) will be handled in the tutorial session on Monday, December 22. You only need to hand in the answer to question d).
Which of the following are true and which are false (in a given flow network)? Justify your answer with a (short) proof or counterexample.

a) Given a maximum flow f, then for all $u, v \in V$ the flow $f(u, v)$ from node u to node v or the flow $f(v, u)$ is 0.

Remark: $f(u, v)$ is defined as 0 if there is no directed edge from u to v.

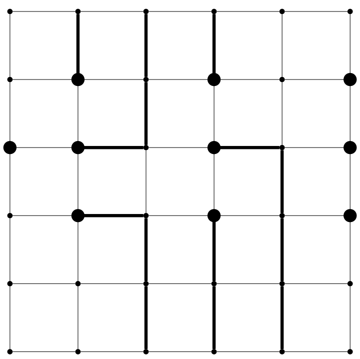
b) Given a maximum flow f, then there exist $u, v \in V$ such that the flow $f(u, v)$ from node u to node v or the flow $f(v, u)$ is 0.

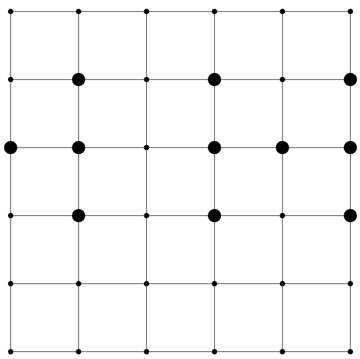
c) For all $u, v \in V$ there exists a maximum flow f, such that the flow $f(u, v)$ from node u to node v or the flow $f(v, u)$ from v to u is 0.

d) There exists a maximum flow f, such that for all $u, v \in V$ the flow $f(u, v)$ from node u to node v or the flow $f(v, u)$ from v to u is 0.

Exercise 1: Circuit Board, Conducting Paths (4 points)

A circuit board with a rectangle grid is given. Some designated points of the grid are connectors and need to be connected with ports via conducting paths. The ports can be anywhere on the rim of the grid. The conducting paths need to run along the grid lines and conducting paths are not allowed to cross.
Design an algorithm which finds a solution for a given grid with designated points or indicates that there is no solution. Your algorithm should run in polynomial time.
Remark 1: If you use any flow network, describe it explicitly.
Remark 2: There is a solution in the left hand side example but no solution in the right hand side example.



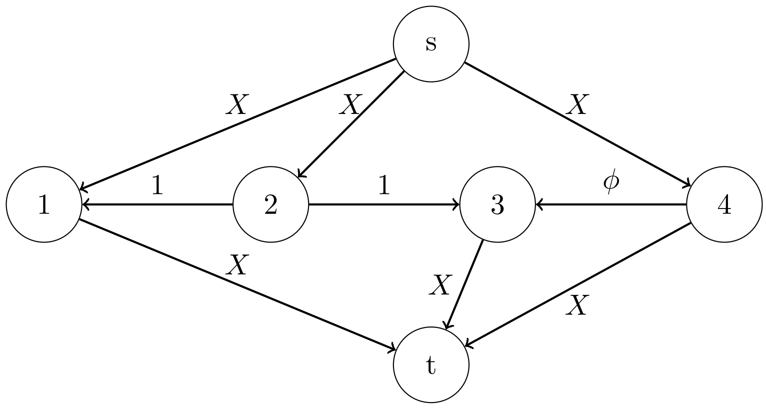


Exercise 2: Non terminating Ford Fulkerson (0.5+0.5+3+2+1 points)

The Ford Fulkerson algorithm, if it terminates, always produces a maximal flow. If edge weights are integral termination is ensured. But it does not need to terminate if the edge weights are allowed to be irrational and the augmenting paths are chosen unsuitably.
The following construction can lead to a flow network in which the Ford Fulkerson algorithm does not terminate: Let $\phi := \frac{1}{2}(\sqrt{5} - 1)$ and let X denote a large (at least 3)[1] integral constant. Note that

$$\phi^n = \underbrace{\phi \cdot \ldots \cdot \phi}_{n} = \phi^{n+1} + \phi^{n+2} \text{ holds for all } n \geqslant 0.$$

Consider the following flow network:



We name the three (directed) horizontal edges between the nodes $1, 2, 3$ and 4 as follows:

$$e_x = (2,1), \;\; e_y = (2,3), \;\; e_z = (4,3).$$

The residual capacities of the forward edges e_x, e_y and e_z corresponding to a flow are denoted by $r(e_x), r(e_y)$ and $r(e_z)$.

a) Show that the value of a maximum flow of the above flow network is $2X + 1$ (assuming that X is large).

b) Starting with an empty flow, show that there is an augmenting path leading to the residual capacities $r(e_x) = \phi^0,\; r(e_y) = 0$ and $r(e_z) = \phi$.

c) Given a flow with the residual capacities in line $i = 1, \ldots, 4$, find an augmenting path (augment the flow with the bottleneck value of the path) to obtain the residual capacities in line $i + 1$:[2]

$$r(e_x) = \phi^k, \qquad r(e_y) = 0, \qquad r(e_z) = \phi^{k+1}, \tag{1}$$
$$r(e_x) = \phi^{k+2}, \qquad r(e_y) = \phi^{k+1}, \qquad r(e_z) = 0, \tag{2}$$
$$r(e_x) = \phi^{k+2}, \qquad r(e_y) = 0, \qquad r(e_z) = \phi^{k+1}, \tag{3}$$
$$r(e_x) = 0, \qquad r(e_y) = \phi^{k+2}, \qquad r(e_z) = \phi^{k+3}, \tag{4}$$
$$r(e_x) = \phi^{k+2}, \qquad r(e_y) = 0, \qquad r(e_z) = \phi^{k+3}. \tag{5}$$

d) Conclude that the Ford Fulkerson algorithm does not terminate on this network if the augmenting paths are not chosen properly. Does the algorithm converge to a value? If so, to which value does it converge?

e) Find a network where the Ford Fulkerson algorithm converges, but not to a maximal flow.

[1] Assume that X is large enough such that you do not need to consider its value during your computation of flows.
[2] One needs to give four augmenting paths here.