Chapter 2
Greedy Algorithms

Algorithm Theory
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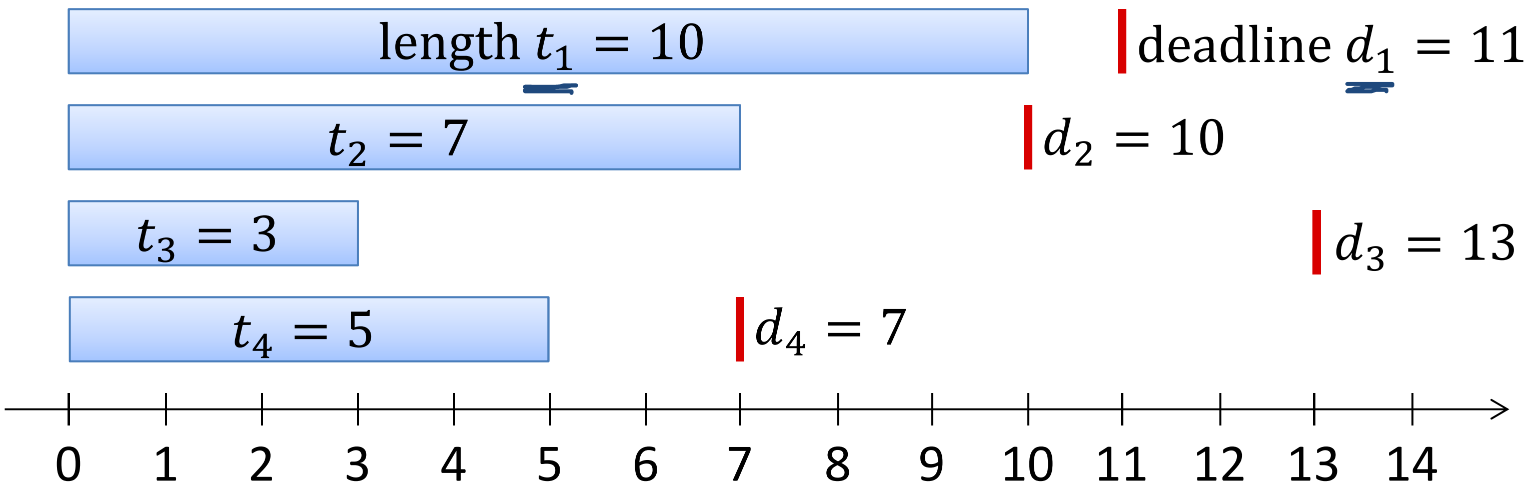
Scheduling Jobs with Deadlines

- Given: n requests / jobs with deadlines:



- Goal: schedule all jobs with minimum lateness L
  - Schedule: $s(i)$, $f(i)$: start and finishing times of request i
    Note: $f(i) = s(i) + t_i$

- Lateness $L := \max\left\{0, \max_i\{f(i) - d_i\}\right\}$
  - largest amount of time by which some job finishes late

- Many other natural objective functions possible…

Greedy Algorithm

Schedule by earliest deadline?

- Schedule in increasing order of d_i
- Ignores lengths of jobs: too simplistic?

- Earliest deadline is optimal!

job i : $s(i)$, $f(i)$

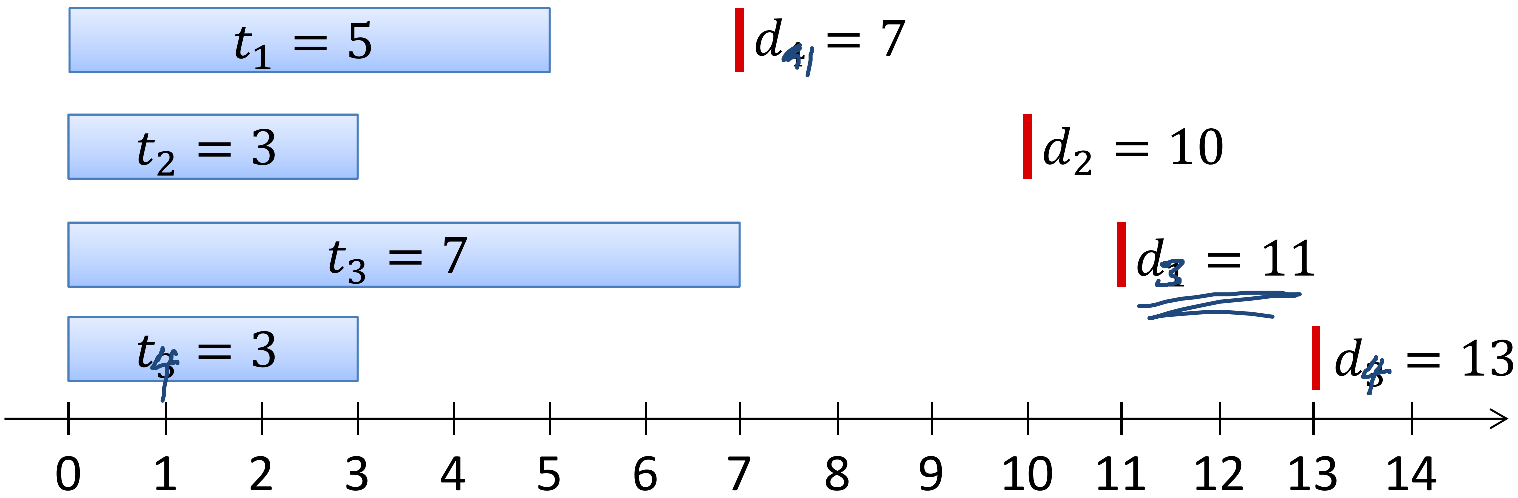
Algorithm:

- Assume jobs are reordered such that $d_1 \leq d_2 \leq \cdots \leq d_n$
- Start/finishing times:
  - First job starts at time $s(1) = 0$
  - Duration of job i is t_i: $f(i) = s(i) + t_i$
  - No gaps between jobs: $s(i+1) = f(i)$

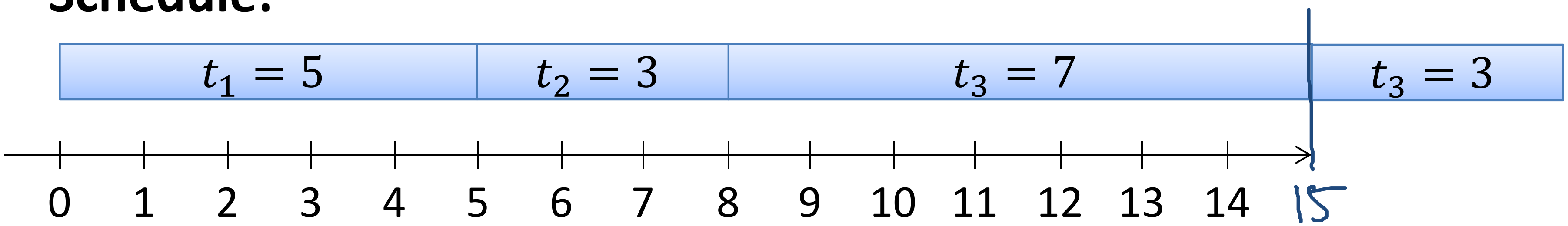
  (idle time: gaps in a schedule → alg. gives schedule with no idle time)

Example

Jobs ordered by deadline:



Schedule:



Lateness: job 1: 0, job 2: 0, job 3: 4, job 4: 5

Basic Facts

1. There is an optimal schedule with no idle time
   - Can just schedule jobs earlier…

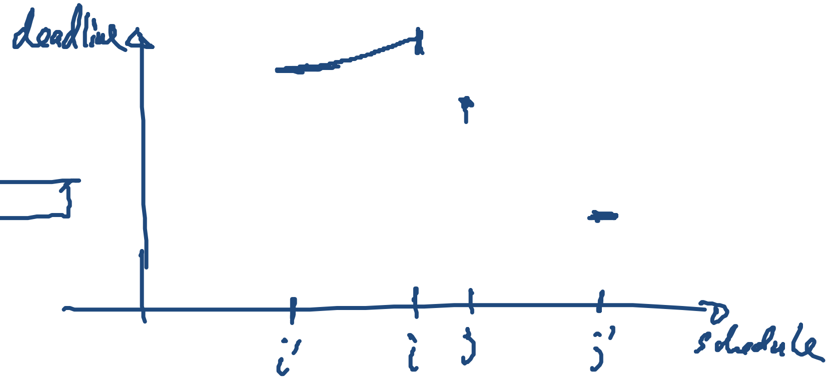
2. Inversion: Job i scheduled before job j if $d_i > d_j$
   Schedules with no inversions have the same maximum lateness
   - In schedules with no inversions, jobs are sorted by deadline
   - Only jobs with the same deadline can be permuted
   - For each deadline d, the maximum lateness remains the same if these jobs are reordered

Earliest Deadline is Optimal

Theorem:
There is an optimal schedule $\mathcal{O}$ with no inversions and no idle time.

Proof: inversion (i', j') $d_{i'} > d_{j'}$ i' sched. before j'

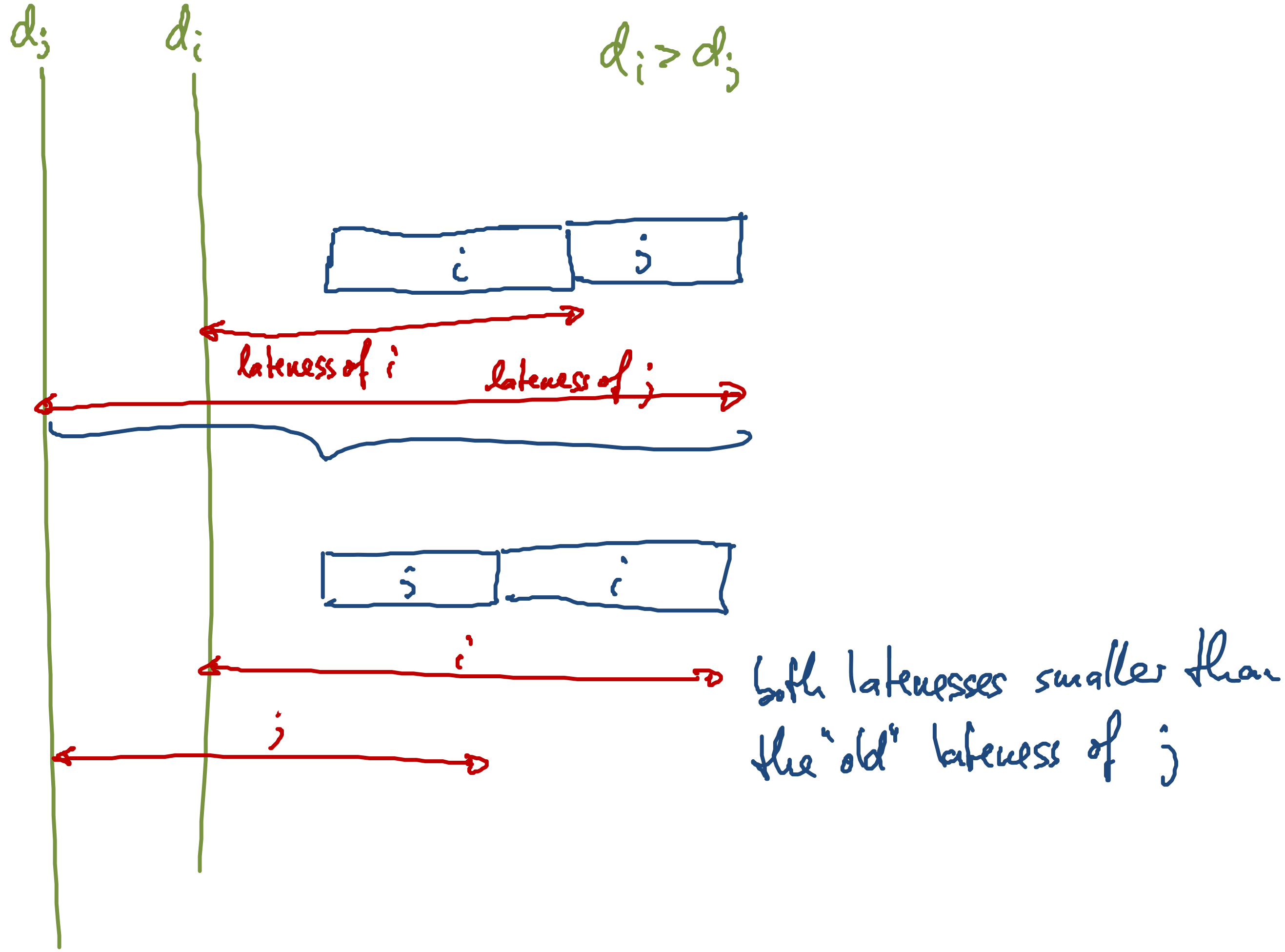
- Consider optimal schedule $\mathcal{O}'$ with no idle time
- If $\mathcal{O}'$ has inversions, $\exists$ pair (i, j), s.t. i is scheduled immediately before j and $d_j < d_i$



- Claim: Swapping i and j gives schedule with
  1. Less inversions
  2. Maximum lateness no larger than in $\mathcal{O}'$

Earliest Deadline is Optimal

Claim: Swapping i and j: maximum lateness no larger than in $\mathcal{O}'$



Exchange Argument

- General approach that often works to analyze greedy algorithms

- Start with any solution
- Define basic exchange step that allows to transform solution into a new solution that is not worse
- Show that exchange step move solution closer to the solution produced by the greedy algorithm
- Number of exchange steps to reach greedy solution should be finite…

Another Exchange Argument Example

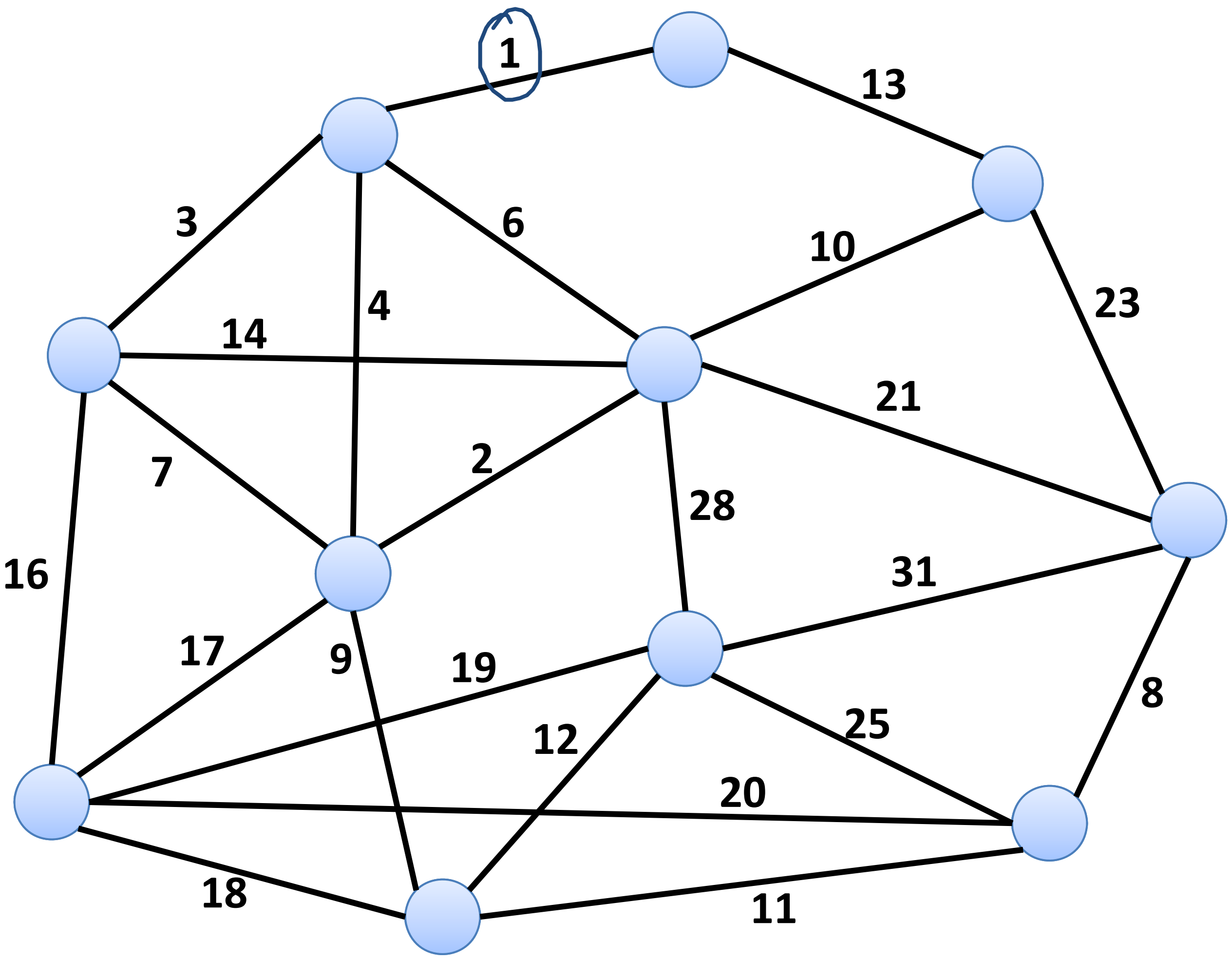
- **Minimum spanning tree (MST)** problem
  - Classic graph-theoretic optimization problem

- **Given**: weighted graph
- **Goal**: spanning tree with min. total weight

- Several greedy algorithms work

- Kruskal's algorithm:
  - Start with empty edge set
  - As long as we do not have a spanning tree:
    add minimum weight edge that doesn't close a cycle

Kruskal Algorithm: Example



Kruskal is Optimal

- Basic exchange step: swap to edges to get from tree T to tree T'
  - Swap out edge not in Kruskal tree, swap in edge in Kruskal tree
  - Swapping does not increase total weight

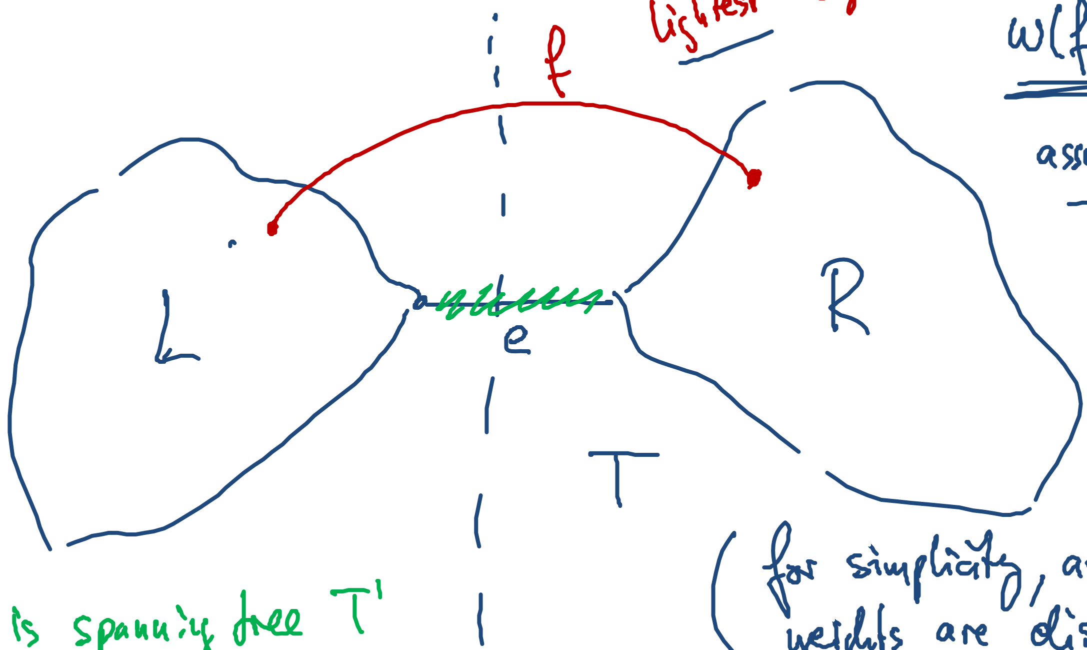
T : any spanning tree T_K : Kruskal tree

$T \neq T_K$ $e \in T \setminus T_K$

f: lightest edge connecting L and R in T_K

$w(f) \leq w(e)$

assume $w(e) < w(f)$ → contradiction



$T \setminus \{e\} \cup \{f\}$ is spanning tree T'

(for simplicity, assume edge weights are distinct)

Matroids

- Same, but more abstract…

Matroid: **pair (E, I)**

- $\boldsymbol{E}$: set, called the **ground set** set of elements
- $\boldsymbol{I}$: finite family of finite subsets of E (i.e., $I \subseteq 2^E$), called **independent sets**

$\forall A \in I\ \forall A' \subseteq A: A' \in I$

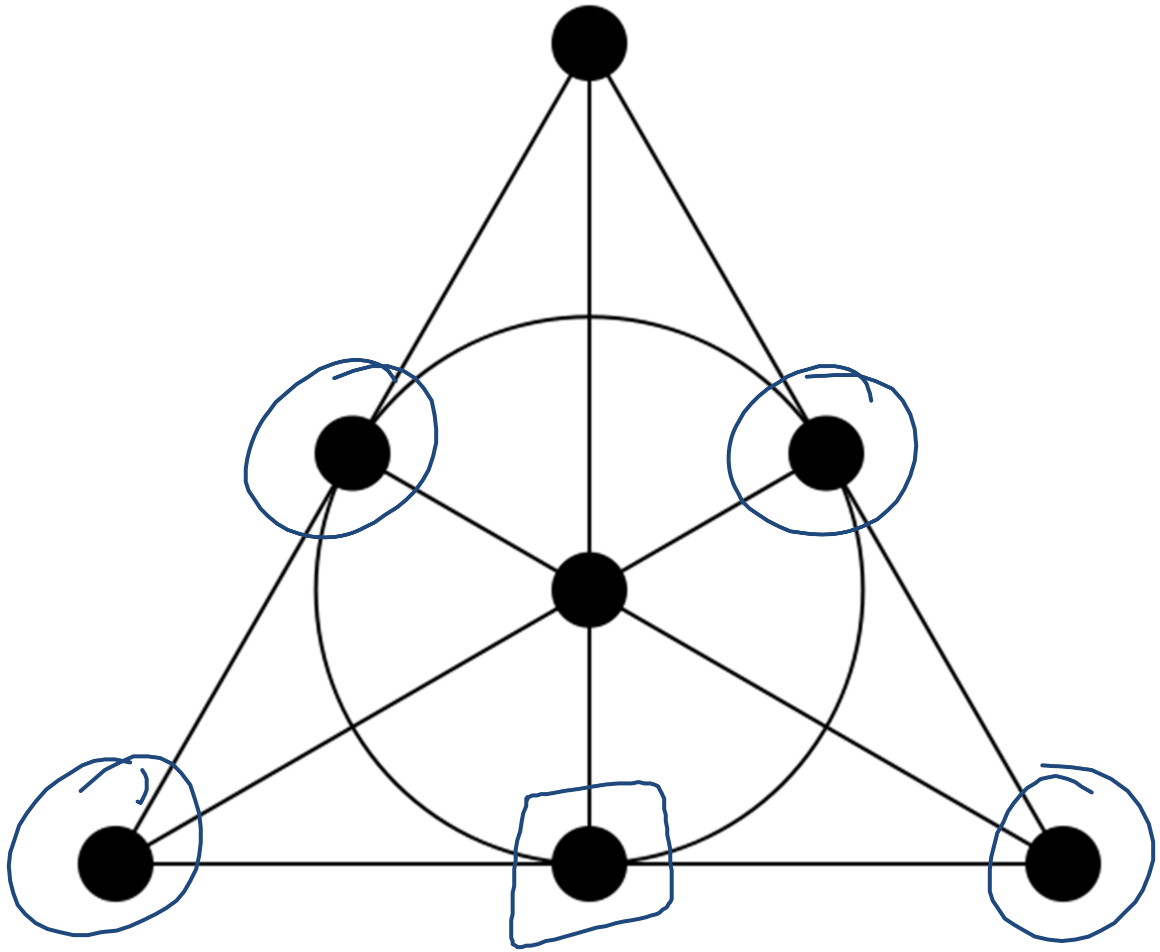
(E, I) needs to satisfy 3 properties:

1. Empty set is independent, i.e., $\emptyset \in I$ (implies that $I \neq \emptyset$)
2. **Hereditary property**: For all $A \subseteq E$ and all $A' \subseteq A$, **if $A \in I$, then also $A' \in I$**
3. **Augmentation / Independent set exchange property:** If $A, B \in I$ and $|A| > |B|$, there exists $x \in A \setminus B$ such that

$$\mathbf{B}' := \boldsymbol{B} \cup \{x\} \in \boldsymbol{I}$$

Example

- Fano matroid:
  - Smallest finite projective plane of order 2…



Matroids and Greedy Algorithms

Weighted matroid: each $e \in E$ has a weight $w(e) > 0$

Goal: find maximum weight independent set

Greedy algorithm:

1. Start with $S = \emptyset$
2. Add max. weight $e \in E \setminus S$ to S such that $S \cup \{e\} \in I$

Claim: greedy algorithm computes optimal solution

Greedy is Optimal

$s := |S|$

$a := |A|$

- S: greedy solution A: any other solution (any ind. set) $a \le s$

$S \subseteq E$ $A \subseteq E$

$|S| \ge |A|$

for contrad., assume $|S| < |A| \xrightarrow{\text{i.s. ex. prop}} \exists x \in A \setminus S : \underbrace{S}_{S'} \cup \{x\} \in \mathcal{I}$

x could have been added

Again for contr. assume $w(S) < w(A)$

$S = \{x_1, x_2, x_3, \dots, x_s\}$ $w(x_1) \ge w(x_2) \dots$

$A = \{y_1, y_2, y_3, \dots, y_a\}$ $w(y_1) \ge w(y_2) \ge \dots$

there has to be a smallest k s.t. $w(x_k) < w(y_k)$

$S' = \{x_1, \dots, x_{k-1}\}$ $\forall i \le k : w(y_i) > w(x_k)$

$A' = \{y_1, \dots, y_k\}$ ex. prop. : greedy adds some $y_i \notin \{x_1, \dots, x_{k-1}\}$

$\Rightarrow$ contradiction