# Chapter 3
# Dynamic Programming

## Algorithm Theory
## WS 2014/15

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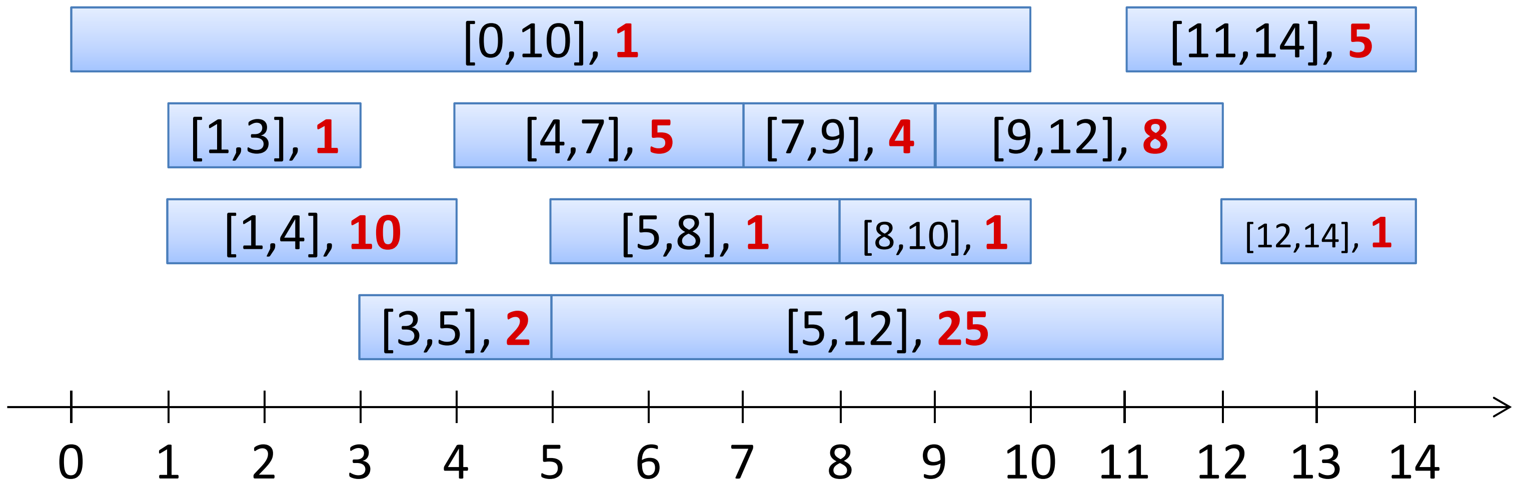
# Weighted Interval Scheduling

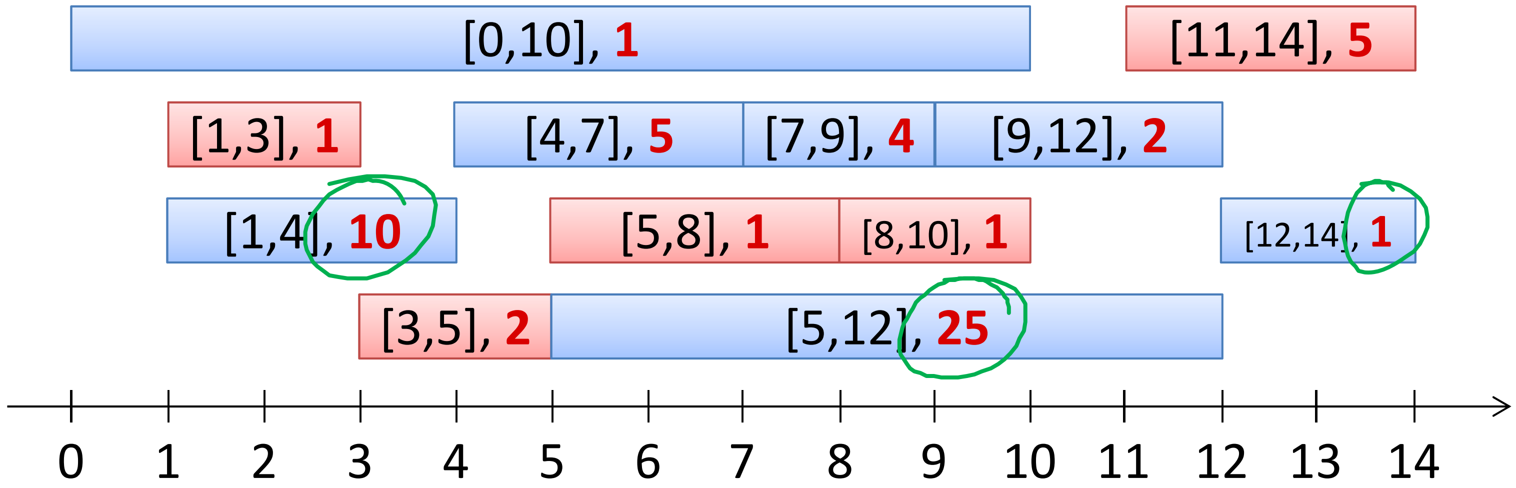
- **Given:** Set of intervals, e.g.
  [0,10],[1,3],[1,4],[3,5],[4,7],[5,8],[5,12],[7,9],[9,12],[8,10],[11,14],[12,14]
- Each interval has a **weight** ***w***




- **Goal:** Non-overlapping set of intervals of largest possible weight
  - Overlap at boundary ok, i.e., [4,7] and [7,9] are non-overlapping
- **Example:** Intervals are room requests of different importance

# Greedy Algorithms

**Choose available request with earliest finishing time:**



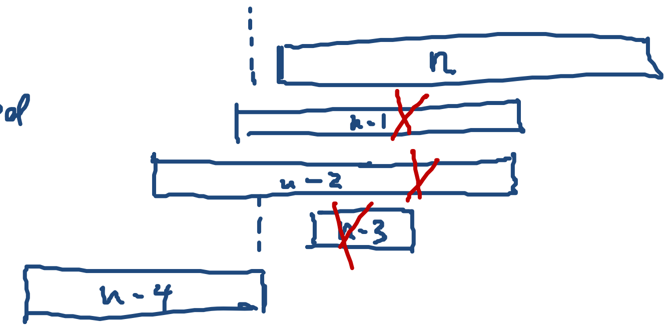

- Algorithm is not optimal any more
  - It can even be arbitrarily bad…

- No greedy algorithm known that works

# Solving Weighted Interval Scheduling

- Interval $i$: start time $s(i)$, finishing time: $f(i)$, weight: $w(i)$
- Assume intervals $1, \dots, n$ are sorted by increasing $f(i)$
  - $0 < f(1) \le f(2) \le \dots \le f(n)$, for convenience: $f(0) = 0$
- Simple observation:
  Opt. solution contains interval $n$ or it doesn't contain interval $n$

- opt. sol. doesn't contain n
  → opt. sol. for int. 1 – n is the same as opt. sol. for int. 1 – (n-1)
- opt. sol. contains int. n
  in the example:
  opt. sol. for int. 1 – n is composed of opt. sol. for int. 1 – (n-4) and interval n



# Solving Weighted Interval Scheduling

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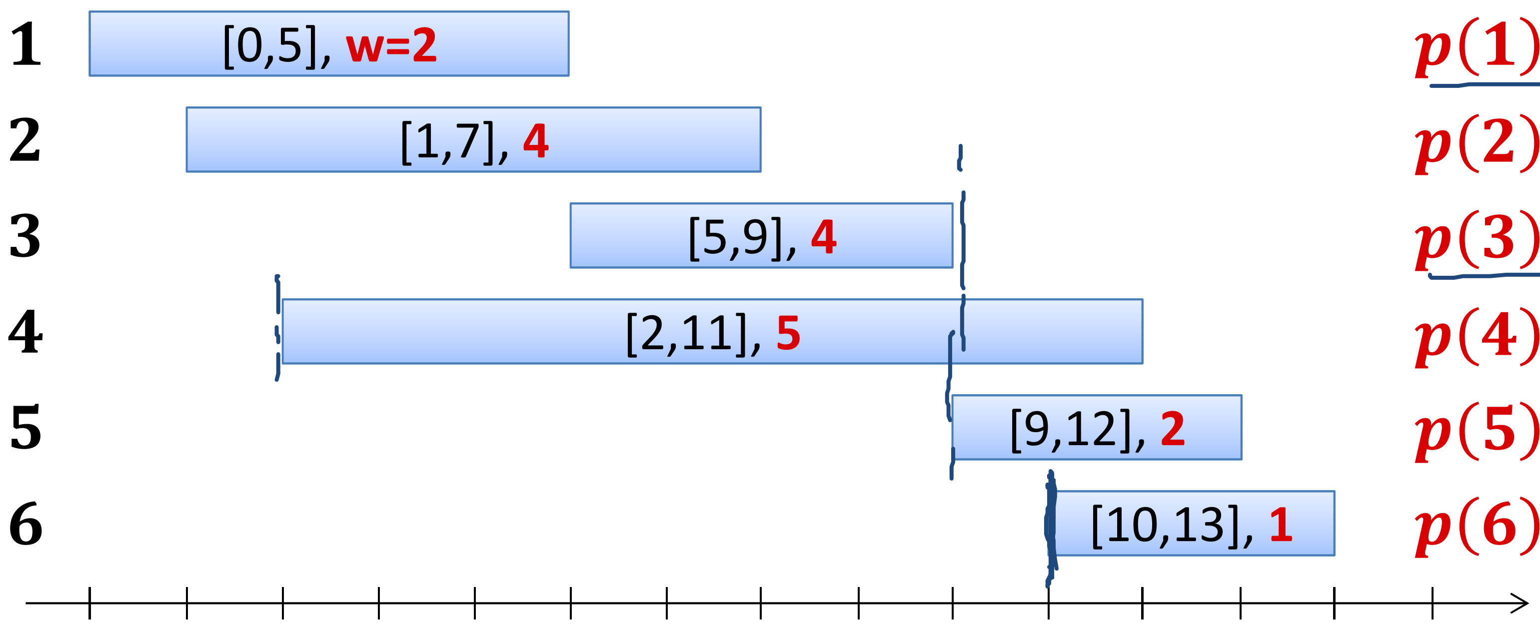
- Assume intervals $1, \dots, n$ are sorted by increasing $f(i)$
  - $0 < f(1) \le f(2) \le \cdots \le f(n)$, for convenience: $f(0) = 0$
- Simple observation:
  Opt. solution contains interval $n$ or it doesn't contain interval $n$
- Weight of optimal solution for only intervals $1, \dots, k$: $W(k)$
  Define $p(k) \coloneqq \max\{i \in \{0, \dots, k-1\} : f(i) \le s(k)\}$
- Opt. solution does not contain interval $n$: $\boldsymbol{W(n) = W(n-1)}$

  Opt. solution contains interval $n$: $\boldsymbol{W(n) = w(n) + W(p(n))}$

in example
$p(n) = n-4$

# Example

**Interval:**




$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 0$

$p(5) = 3$

$p(6) = 3$

Computing $p(k)$: binary search

Compute all $p(k)$ in $O(n \log n)$ time

# Recursive Definition of Optimal Solution

- Recall:
  - $W(k)$: weight of optimal solution with intervals $1, \dots, k$
  - $p(k)$: last interval to finish before interval $k$ starts

- Recursive definition of optimal weight:

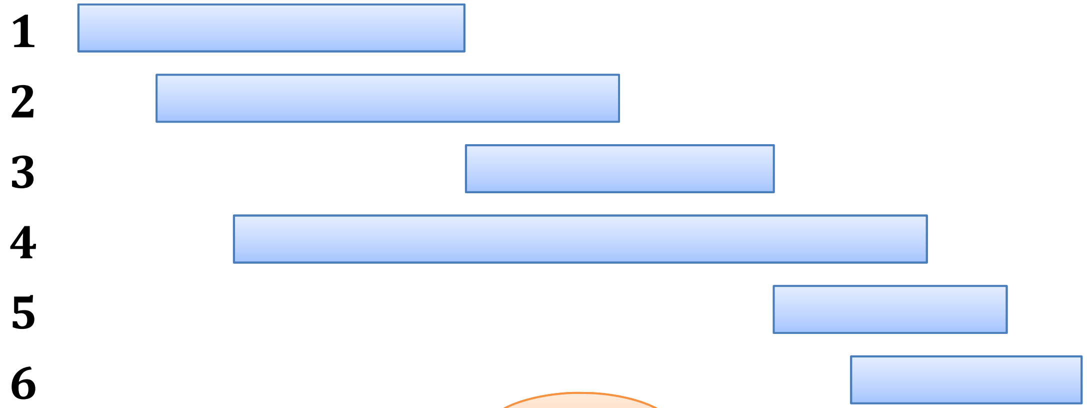
$$\forall k > 1:\; W(k) = \max\{\underline{W(k-1)}, \underline{w(k) + W(p(k))}\}$$

$$W(1) = w(1)$$

goal: find $W(n)$

- Immediately gives a simple, recursive algorithm

# Running Time of Recursive Algorithm




$p(1) = \mathbf{0}$

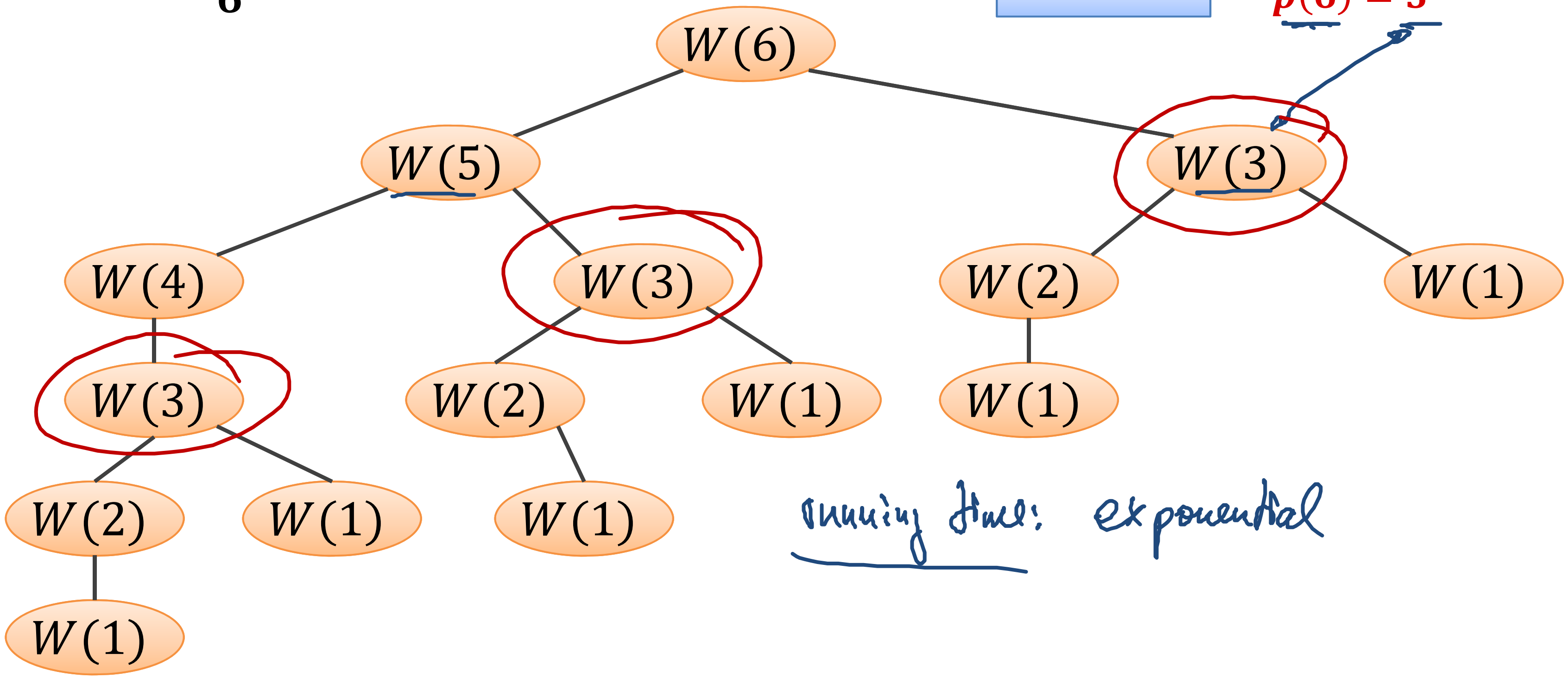
$p(2) = \mathbf{0}$

$p(3) = \mathbf{1}$

$p(4) = \mathbf{0}$

$p(5) = \mathbf{3}$

$p(6) = \mathbf{3}$




running time: exponential

# Memoizing the Recursion

- Running time of recursive algorithm: exponential!
- But, alg. only solves $n$ different sub-problems: $W(1), \dots, W(n)$
- There is no need to compute them multiple times

**Memoization:**

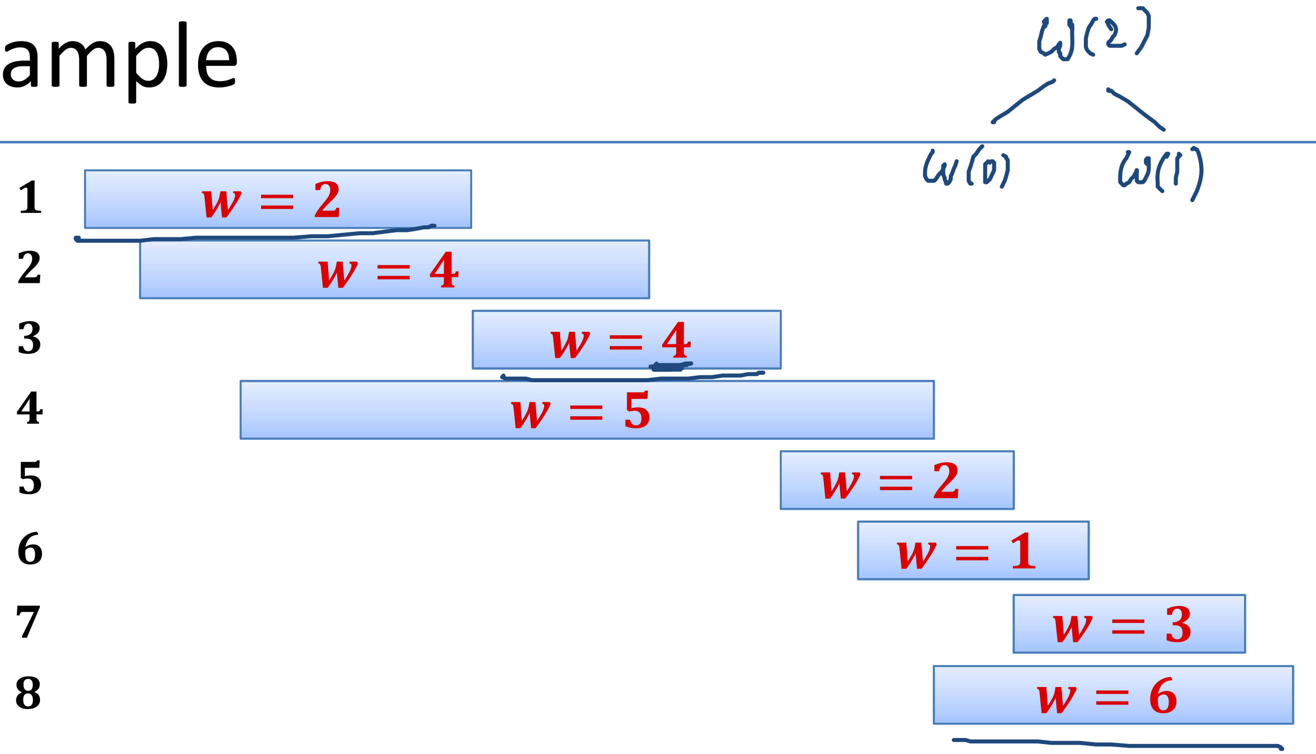
- **Store already computed values** for future use (recursive calls)

**Efficient algorithm:**

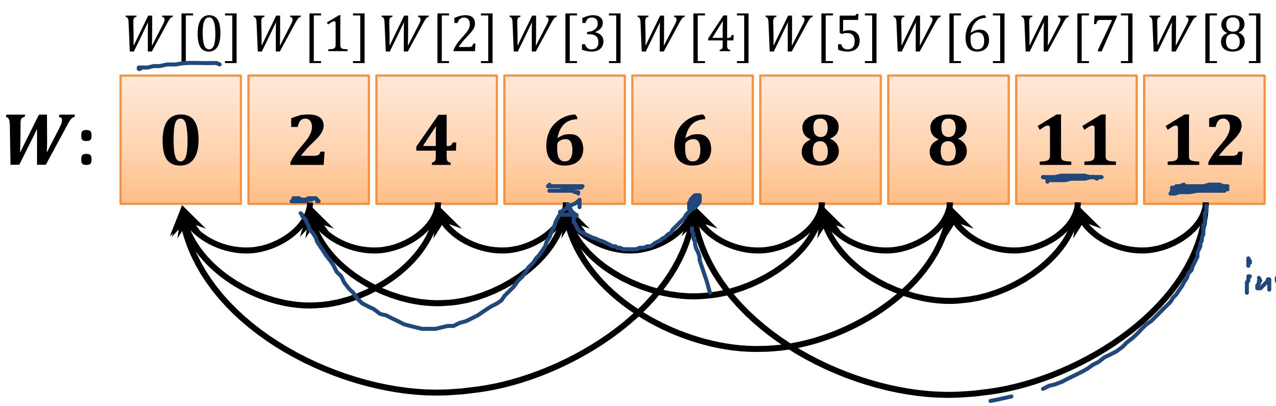
1. $W[0] \coloneqq 0$; compute values $p(i)$
2. **for** $i \coloneqq 1$ **to** $n$ **do**
3. $\quad W[i] \coloneqq \max\{W[i-1], w(i) + W[p(i)]\}$
4. **end**

running time: $O(n)$

# Example




$p(1) = 0$
$p(2) = 0$
$p(3) = 1$
$p(4) = 0$
$p(5) = 3$
$p(6) = 3$
$p(7) = 5$
$p(8) = 4$




int. 1, 3, 8

**Computing the schedule:** **store where you come from!**

# Matrix-chain multiplication

**Given:** sequence (chain) $\langle A_1, A_2, \ldots, A_n \rangle$ of matrices

**Goal:** compute the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$

**Problem:** Parenthesize the product in a way that minimizes the number of scalar multiplications.

**Definition:** A product of matrices is *fully parenthesized* if it is

- a single matrix
- or the product of two fully parenthesized matrix products, surrounded by parentheses.

# Example

All possible fully parenthesized matrix products of the chain $\langle A_1, A_2, A_3, A_4 \rangle$:

$$(A_1(A_2(A_3A_4)))$$

$$(A_1((A_2A_3)A_4))$$

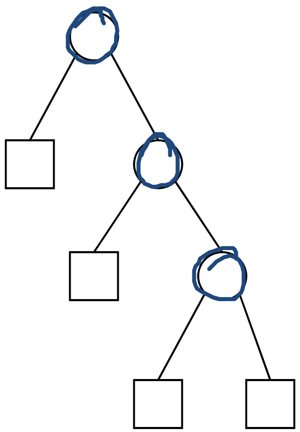
$$((A_1A_2)(A_3A_4))$$

$$((A_1(A_2A_3))A_4)$$

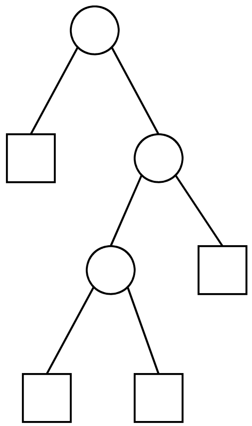
$$(((A_1A_2)A_3)A_4)$$

# Different parenthesizations

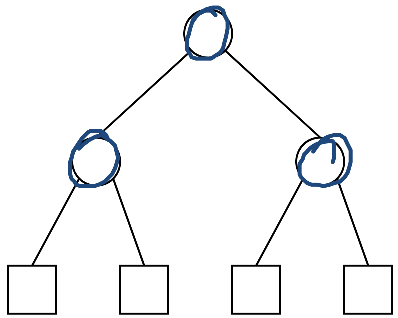
Different parenthesizations correspond to different trees:



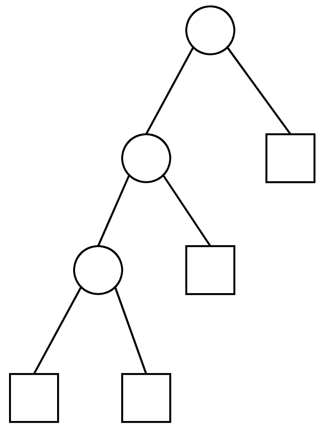
$(A_1(A_2(A_3A_4)))$



$(A_1((A_2A_3)A_4))$



$((A_1A_2)(A_3A_4))$



$(((A_1A_2)A_3)A_4)$

# Number of different parenthesizations

- Let $P(n)$ be the number of alternative parenthesizations of the product $A_1 \cdot \ldots \cdot A_n$:

$$P(1) = 1$$

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k), \qquad \text{for } n \geq 2$$

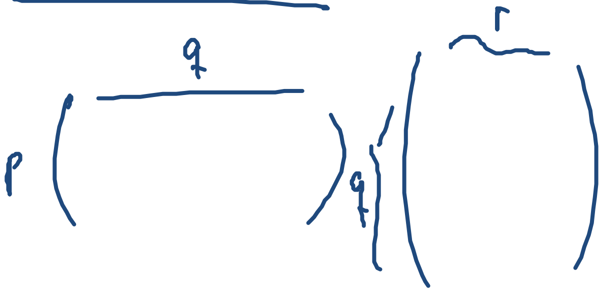
$$P(n+1) = \frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n\sqrt{\pi n}} + O\left(\frac{4^n}{\sqrt{n^5}}\right)$$

$$P(n+1) = C_n \qquad (n^{th} \text{ Catalan number})$$

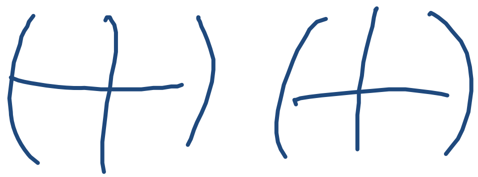
- Thus: Exhaustive search needs exponential time!

# Multiplying Two Matrices

$$A = (a_{ij})_{p\times q}, \qquad B = (b_{ij})_{q\times r}, \qquad A\cdot B = C = (c_{ij})_{p\times r}$$




$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$$



**Remark:**

Using this algorithm, multiplying two $(n \times n)$ matrices requires $n^3$ multiplications. This can also be done using $O(n^{2.376})$ multiplications.

**Algorithm** *Matrix-Mult*

**Input:** $(p \times q)$ matrix $A$, $(q \times r)$ matrix $B$

**Output:** $(p \times r)$ matrix $C = A \cdot B$

**for** $i := 1$ **to** $p$ **do**
  **for** $j := 1$ **to** $r$ **do**
    $C[i,j] := 0;$
    **for** $k := 1$ **to** $q$ **do**
      $C[i,j] := C[i,j] + A[i,k] \cdot B[k,j]$

Number of multiplications and additions: $\boxed{p \cdot q \cdot r}$

# Matrix-chain multiplication: Example

Computation of the product $A_1 A_2 A_3$, where

$A_1$ : $(50 \times 5)$ matrix

$A_2$ : $(5 \times 100)$ matrix

$A_3$ : $(100 \times 10)$ matrix

a) Parenthesization $((A_1 A_2)A_3)$ and $(A_1(A_2 A_3))$ require:

$A' = \underbrace{(A_1 A_2)}_{50\times 100\text{-matrix}}$: $50 \cdot 5 \cdot 100 = 25000$

$A'' = \underbrace{(A_2 A_3)}_{5\times 10\text{-matrix}}$: $5 \cdot 100 \cdot 10 = 5000$

$A' A_3$: $50 \cdot 100 \cdot 10 = 50'000$

$A_1 A''$: $50 \cdot 5 \cdot 10 = 2500$

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Sum: 75'000 | 7'500

# Structure of an Optimal Parenthesization

- $(A_{\ell\ldots r})$: optimal parenthesization of $A_\ell \cdot \ldots \cdot A_r$

  For some $1 \leq k < n$: $(\boldsymbol{A_{1\ldots n}}) = \big((\boldsymbol{A_{1\ldots k}}) \cdot (\boldsymbol{A_{k+1\ldots n}})\big)$

- Any optimal solution contains optimal solutions for sub-problems

- Assume matrix $A_i$ is a $(d_{i-1} \times d_i)$-matrix

- Cost to solve sub-problem $A_\ell \cdot \ldots \cdot A_r, \ell \leq r$ optimally: $C(\ell, r)$

- Then:

$$\boldsymbol{C(a,b) = \min_{a \leq k < b} C(a,k) + C(k+1,b) + d_{a-1} d_k d_b}$$

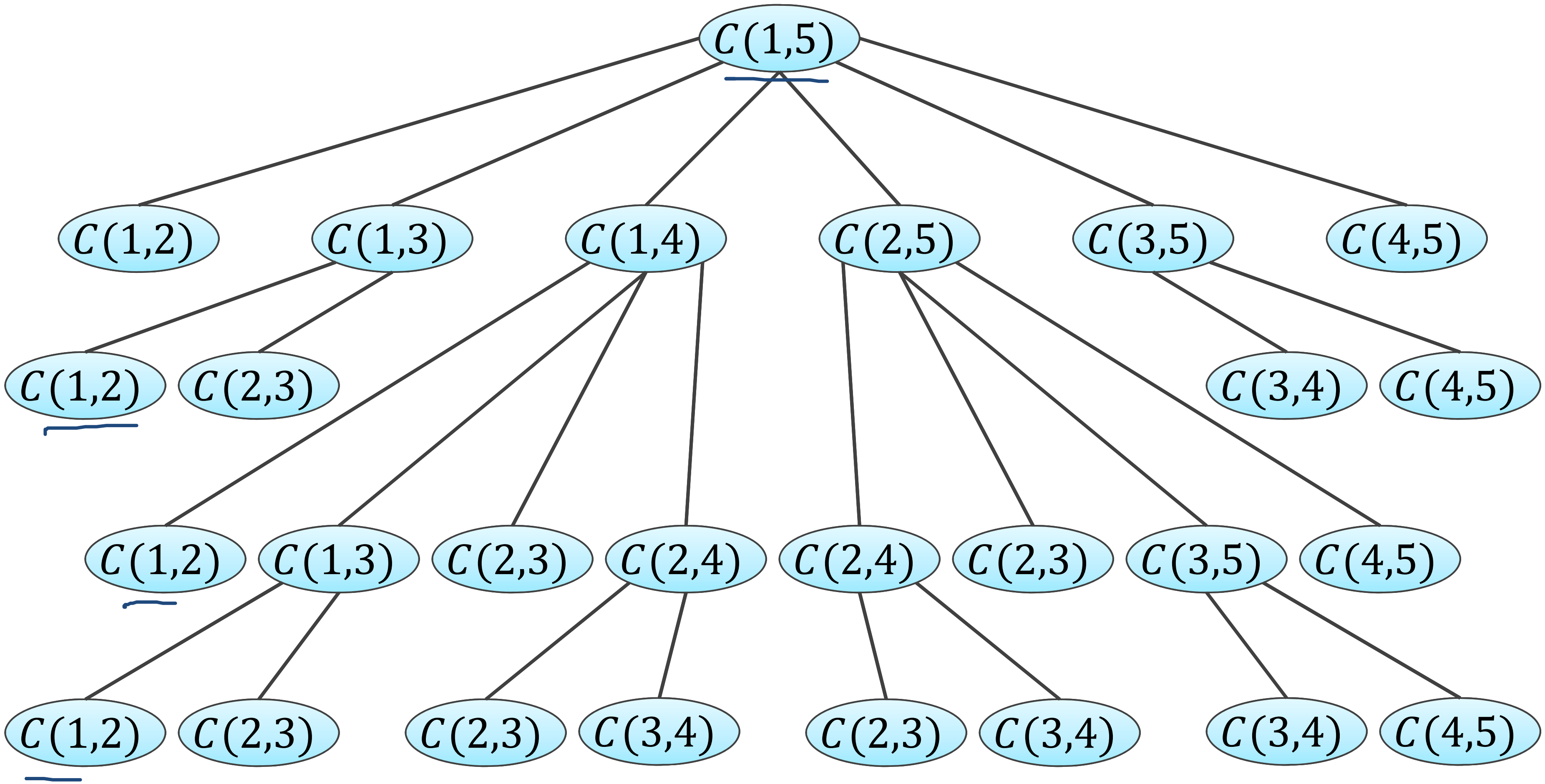
$$\boldsymbol{C(a,a) = 0}$$

we need: $C(1,n)$

$(a \ldots k)(k+1 \ldots b)$

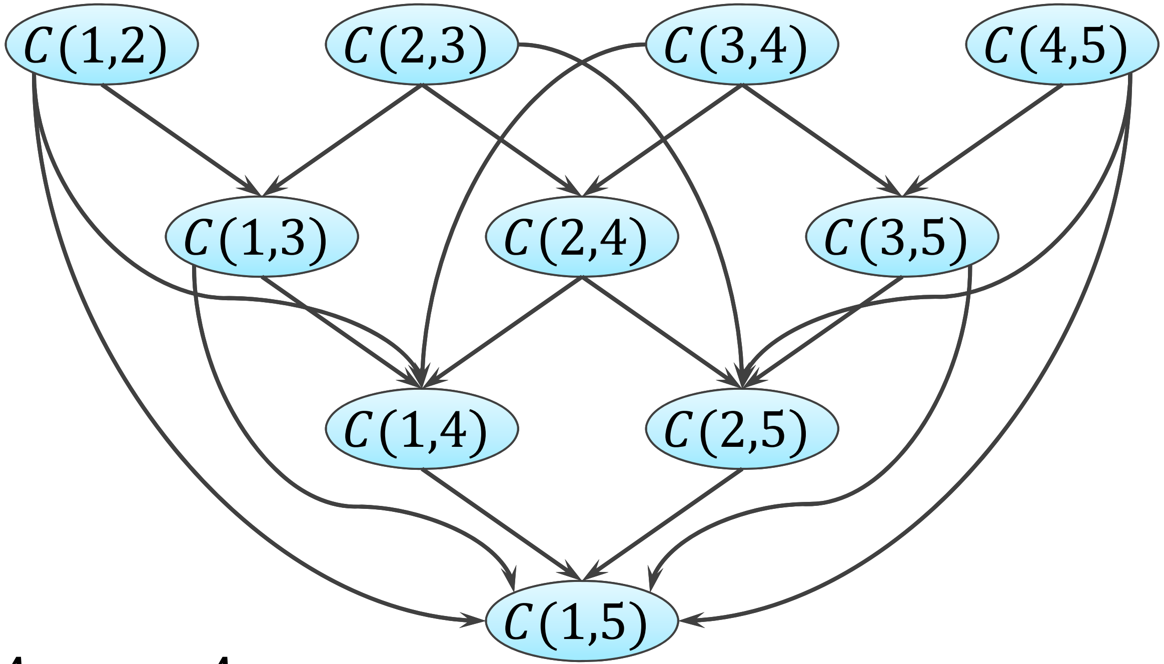
# Recursive Computation of Opt. Solution

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:



# Using Meomization $C(1,2) = d_0 \cdot d_1 \cdot d_2$

Compute $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$:




Compute $A_1 \cdot \ldots \cdot A_n$:

- Each $C(i,j), i < j$ is computed exactly once → $O(n^2)$ values
- Each $C(i,j)$ dir. depends on $C(i,k)$, $C(k+1,j)$ for $i < k < j$

Cost for each $C(i,j)$: $O(n)$ → overall time: $\boldsymbol{O(n^3)}$

# Dynamic Programming

*„Memoization“* for increasing the efficiency of a recursive solution:

- Only the *first time* a sub-problem is encountered, its solution is computed and then stored in a table. Each subsequent time that the subproblem is encountered, the value stored in the table is simply looked up and returned
(without repeated computation!).

- *Computing the solution*: For each sub-problem, store how the value is obtained (according to which recursive rule).

# Dynamic Programming

Dynamic programming / memoization can be applied if

- Optimal solution contains optimal solutions to sub-problems (recursive structure)

- Number of sub-problems that need to be considered is small

# Remarks about matrix-chain multiplication

1. There is an algorithm that determines an optimal parenthesization in time

$$\boxed{O(n \cdot \log n).}$$

2. There is a linear time algorithm that determines a parenthesization using at most

$$1.155 \cdot C(1, n)$$

multiplications.