



Chapter 5

Graph Algorithms

Algorithm Theory
WS 2014/15

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Network Flow: Definition

Flow: function $f: E \rightarrow \mathbb{R}_{\geq 0}$

- $f(e)$ is the amount of flow carried by edge e

Capacity Constraints:

- For each edge $e \in E$, $f(e) \leq c_e$

Flow Conservation:

- For each node $v \in V \setminus \{s, t\}$,

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

Flow Value:

$$|f| := \sum_{e \text{ out of } s} f((s, u)) = \sum_{e \text{ into } t} f((v, t))$$

Notation

We define:

$$f^{\text{in}}(v) := \sum_{e \text{ into } v} f(e), \quad f^{\text{out}}(v) := \sum_{e \text{ out of } v} f(e)$$

For a set $S \subseteq V$:

$$f^{\text{in}}(S) := \sum_{e \text{ into } S} f(e), \quad f^{\text{out}}(S) := \sum_{e \text{ out of } S} f(e)$$

Flow conservation: $\forall v \in V \setminus \{s, t\}: f^{\text{in}}(v) = f^{\text{out}}(v)$

Flow value: $|f| = f^{\text{out}}(s) = f^{\text{in}}(t)$

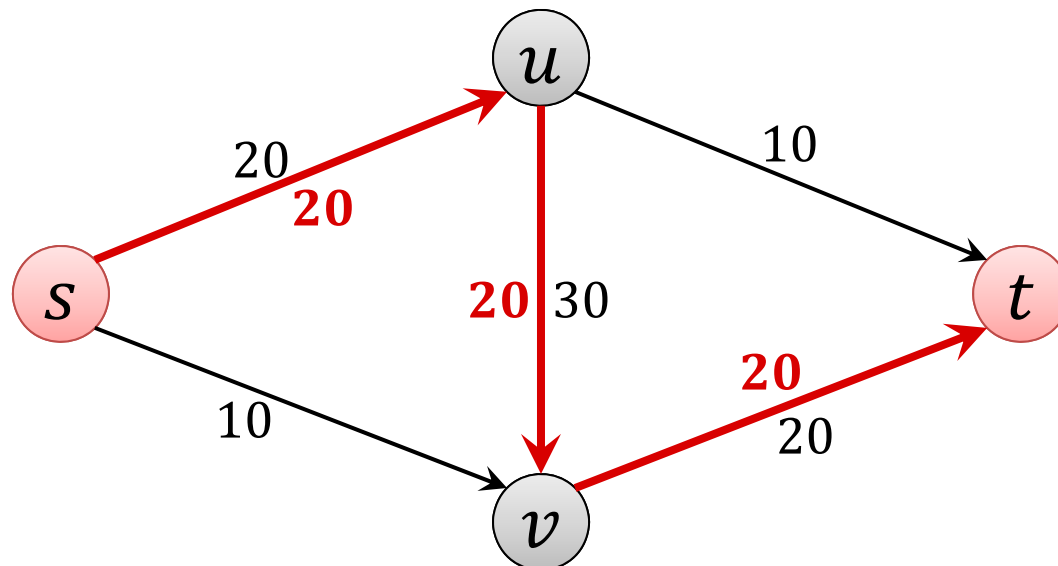
For simplicity: Assume that all **capacities** are **positive integers**

Residual Graph

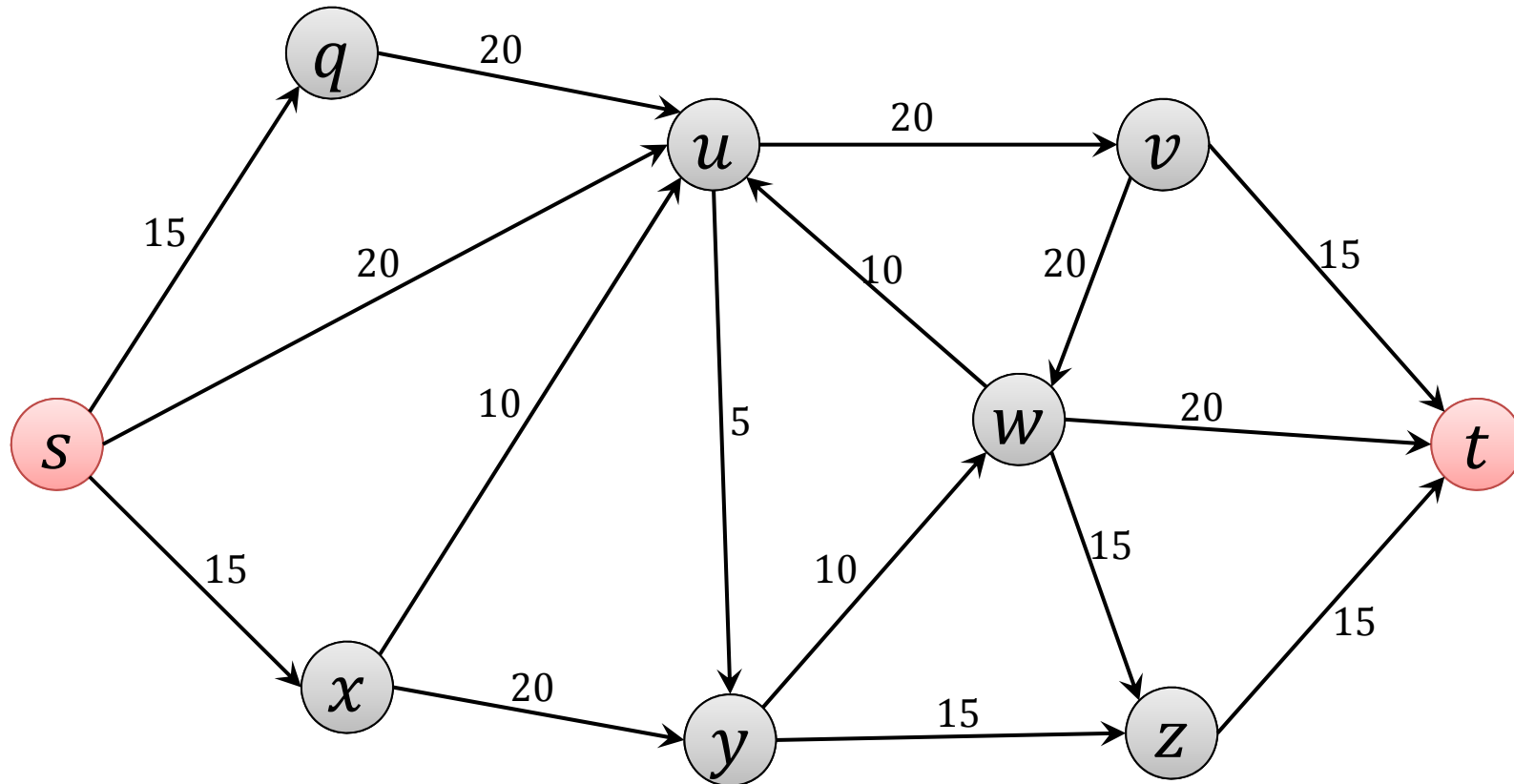
Given a flow network $G = (V, E)$ with capacities c_e (for $e \in E$)

For a flow f on G , define **directed graph** $G_f = (V_f, E_f)$ as follows:

- Node set $V_f = V$
- For each edge $e = (u, v)$ in E , there are two edges in E_f :
 - **forward edge** $e = (u, v)$ with **residual capacity** $c_e - f(e)$
 - **backward edge** $e' = (v, u)$ with **residual capacity** $f(e)$

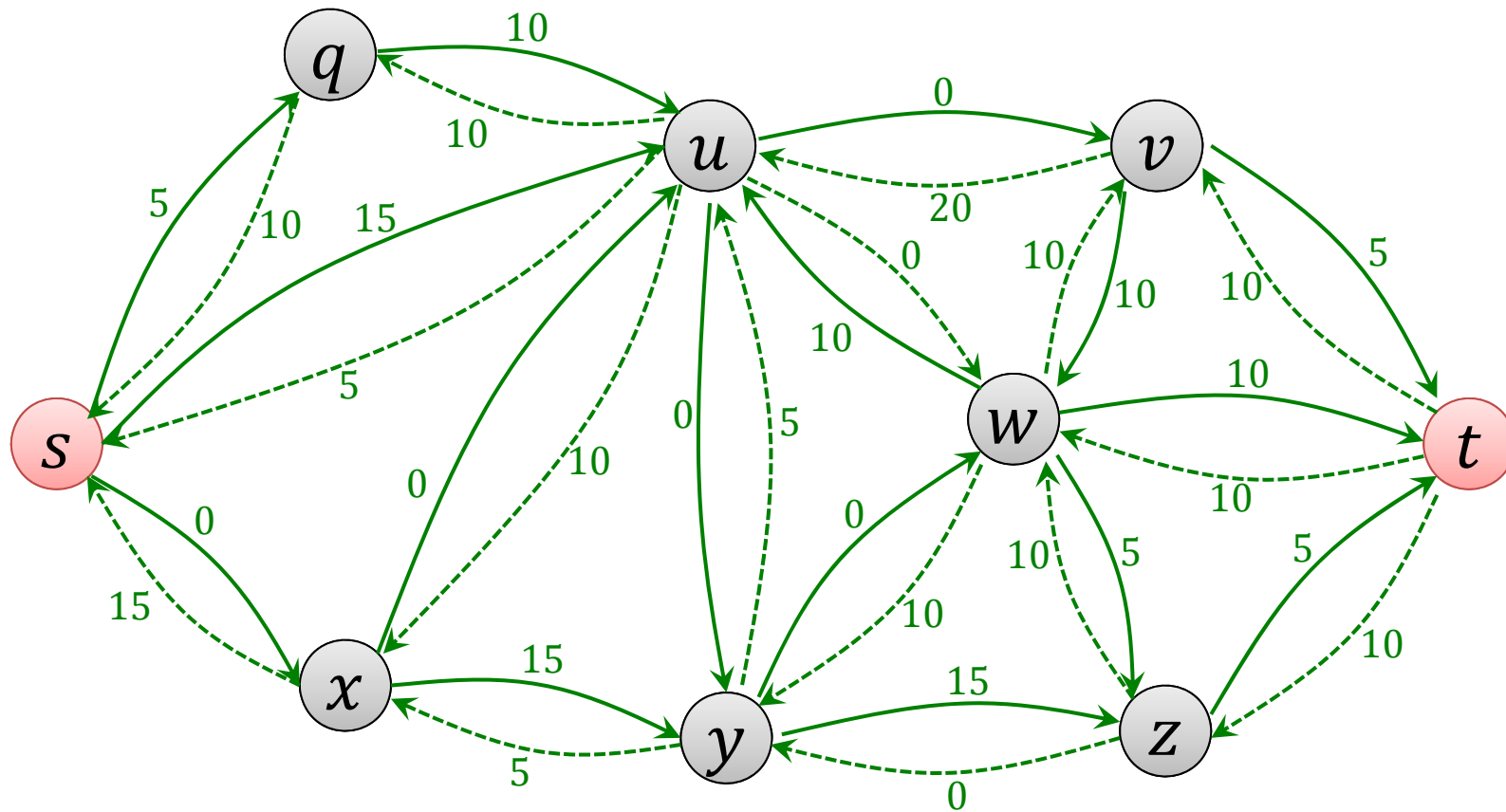


Residual Graph: Example



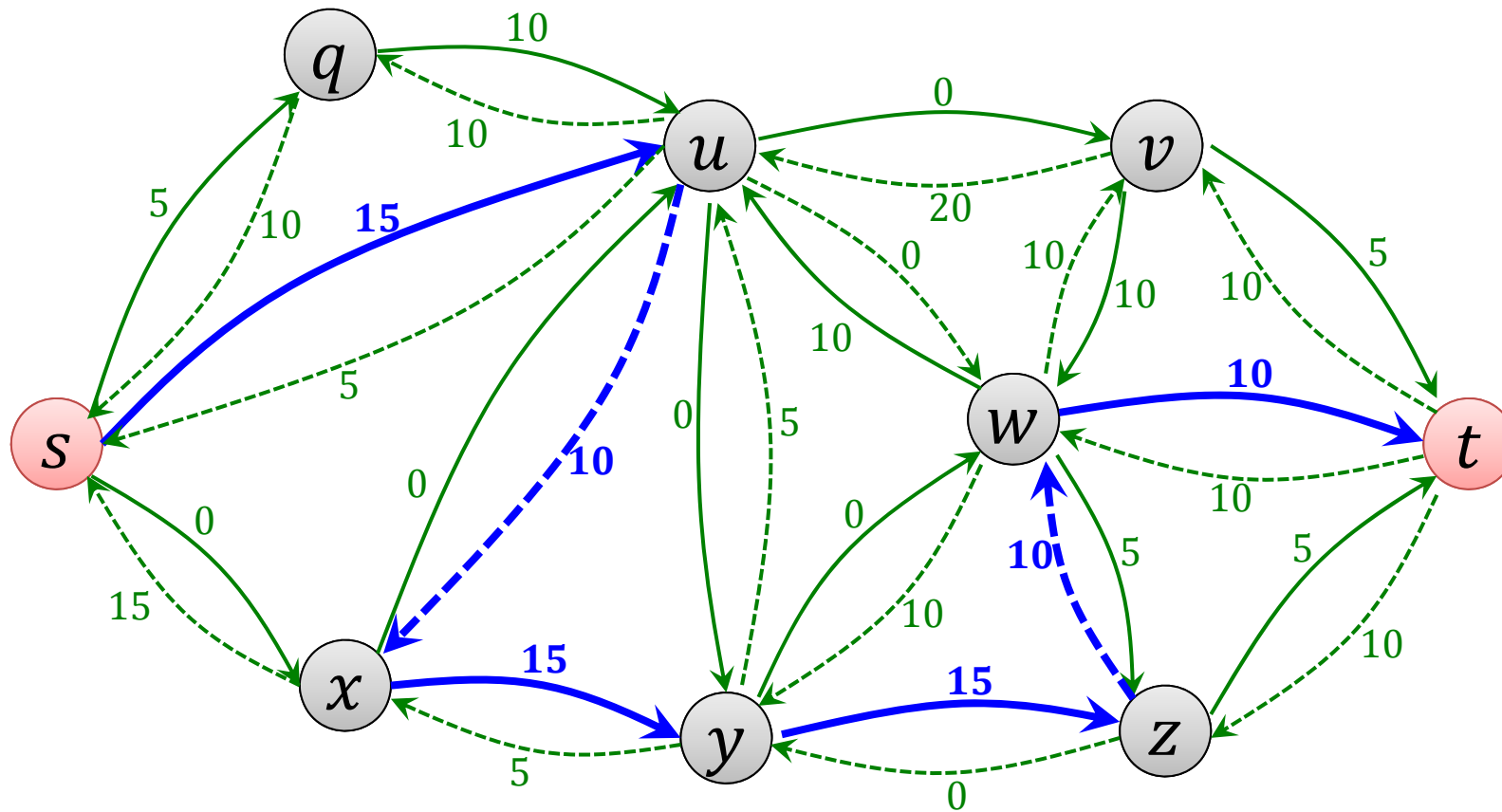
Residual Graph: Example

Residual Graph G_f



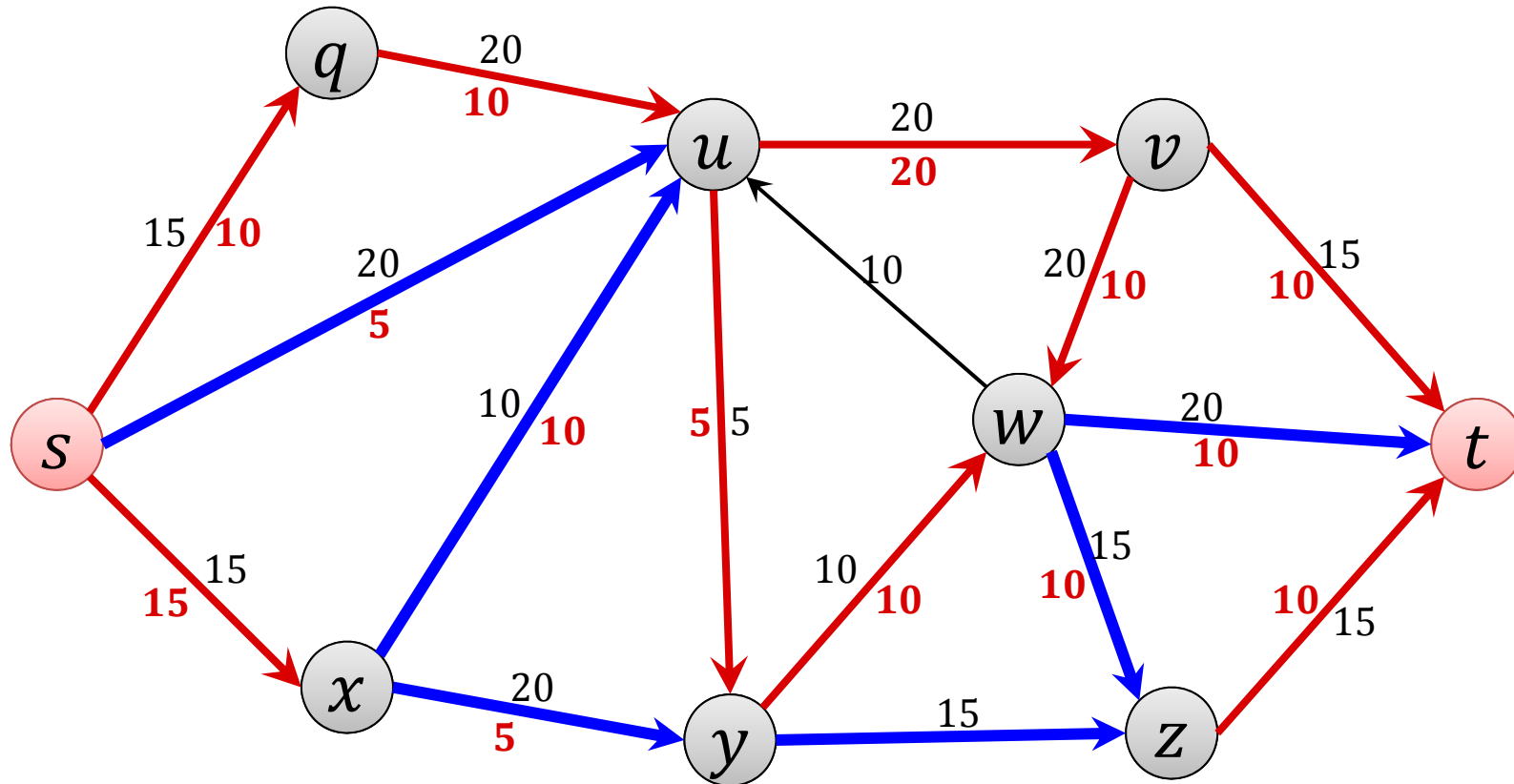
Augmenting Path

Residual Graph G_f



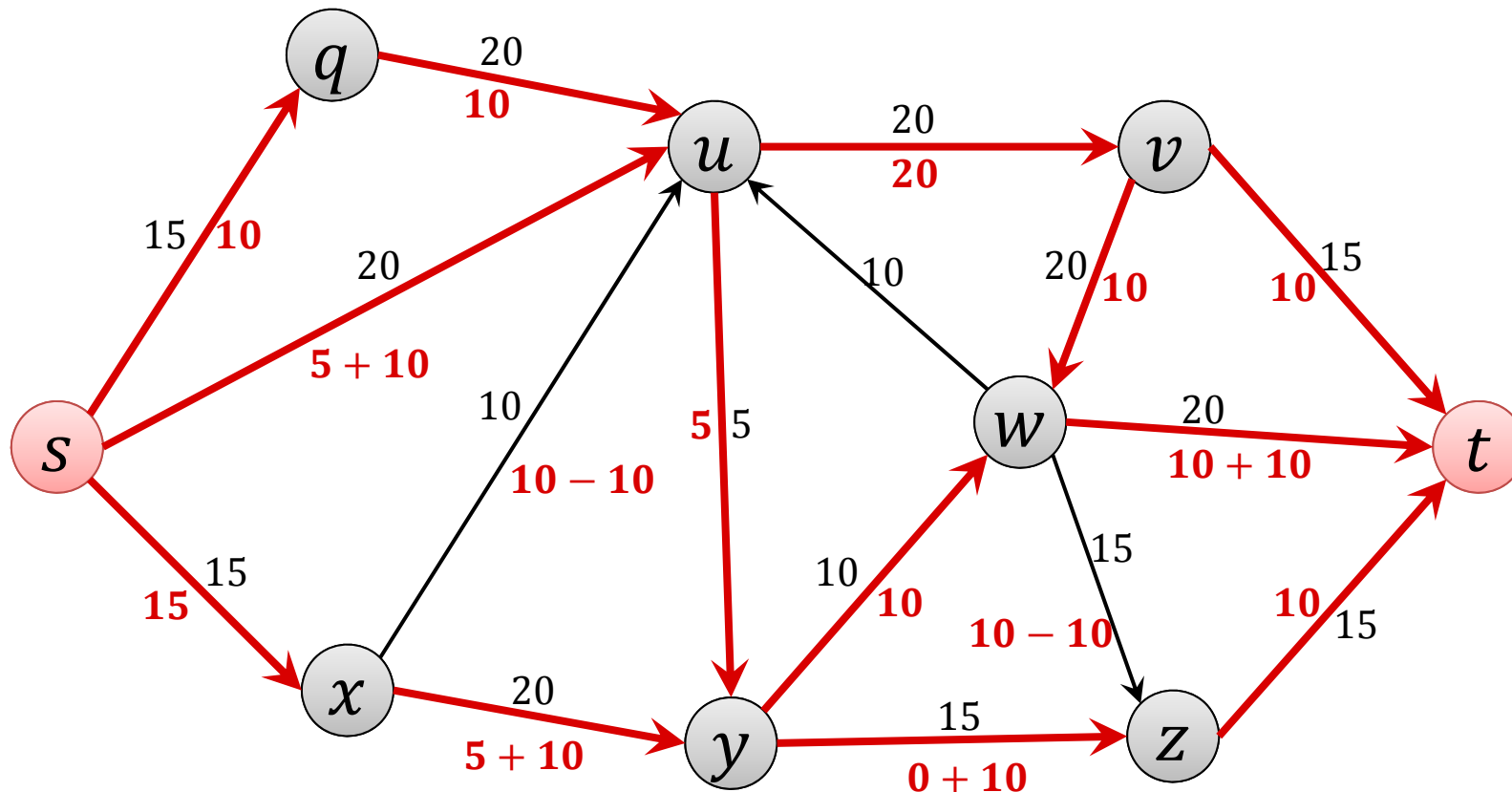
Augmenting Path

Augmenting Path



Augmenting Path

New Flow



Augmenting Path

Definition:

An **augmenting path** P is a (simple) s - t -path on the **residual graph** G_f on which each edge has **residual capacity** > 0 .

bottleneck (P, f) : minimum residual capacity on any edge of the augmenting path P

Augment flow f to get flow f' :

- For every **forward edge** (u, v) on P :

$$f'((u, v)) := f((u, v)) + \mathbf{bottleneck}(P, f)$$

- For every **backward edge** (u, v) on P :

$$f'((v, u)) := f((v, u)) - \mathbf{bottleneck}(P, f)$$

Augmented Flow

Lemma: Given a flow f and an augmenting path P , the resulting augmented flow f' is legal and its value is

$$|f'| = |f| + \mathbf{bottleneck}(P, f).$$

Proof:

Ford-Fulkerson Algorithm

- Improve flow using an augmenting path as long as possible:
 1. Initially, $f(e) = 0$ for all edges $e \in E$, $G_f = G$
 2. **while** there is an augmenting s - t -path P in G_f **do**
 3. Let P be an augmenting s - t -path in G_f ;
 4. $f' := \text{augment}(f, P)$;
 5. update f to be f' ;
 6. update the residual graph G_f
 7. **end**;

Ford-Fulkerson Running Time

Theorem: If all edge capacities are integers, the Ford-Fulkerson algorithm terminates after at most C iterations, where

$$C = \sum_{e \text{ out of } s} c_e.$$

Proof:

Ford-Fulkerson Running Time

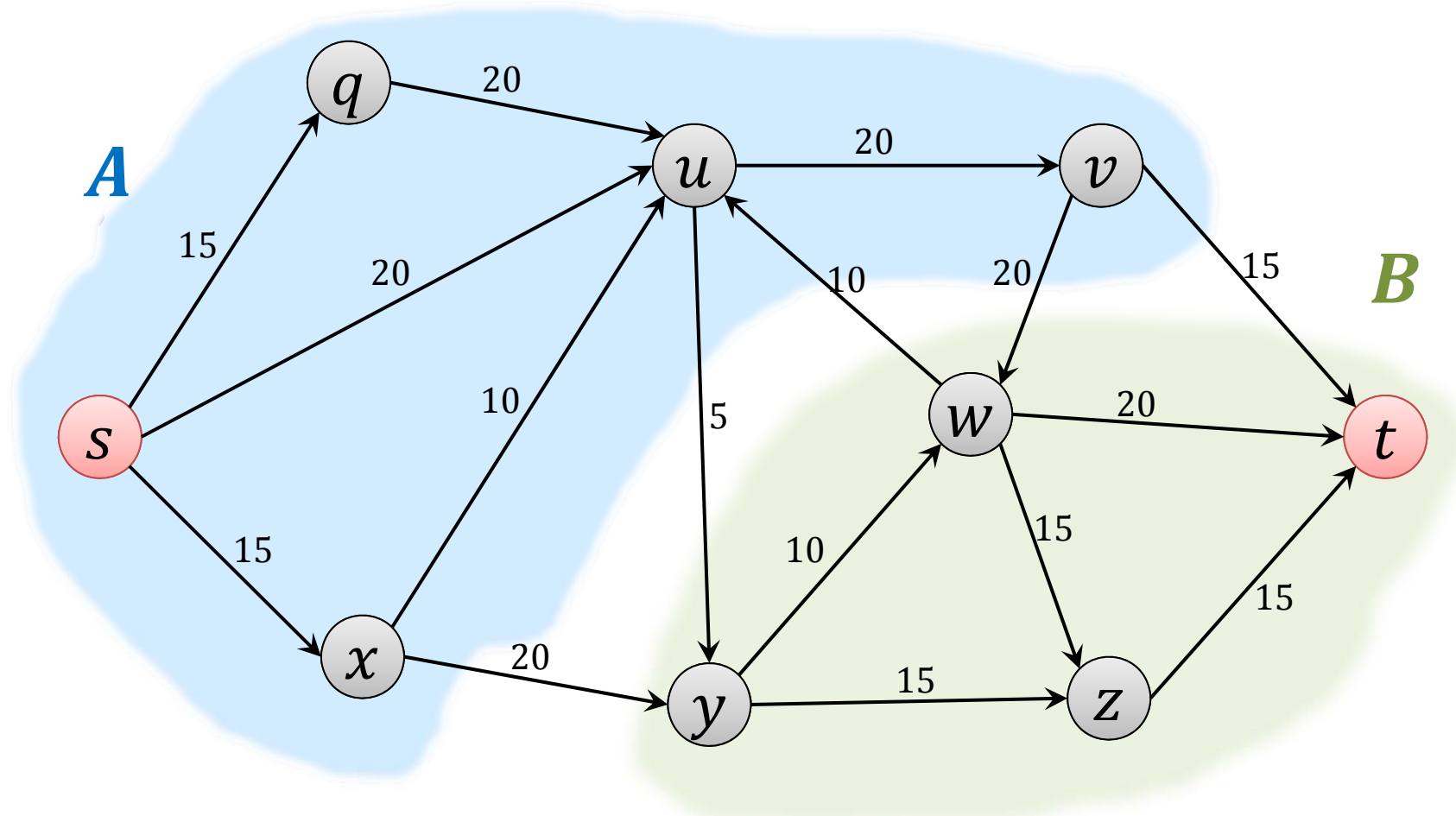
Theorem: If all edge capacities are integers, the Ford-Fulkerson algorithm can be implemented to run in $O(mC)$ time.

Proof:

s - t Cuts

Definition:

An s - t cut is a partition (A, B) of the vertex set such that $s \in A$ and $t \in B$

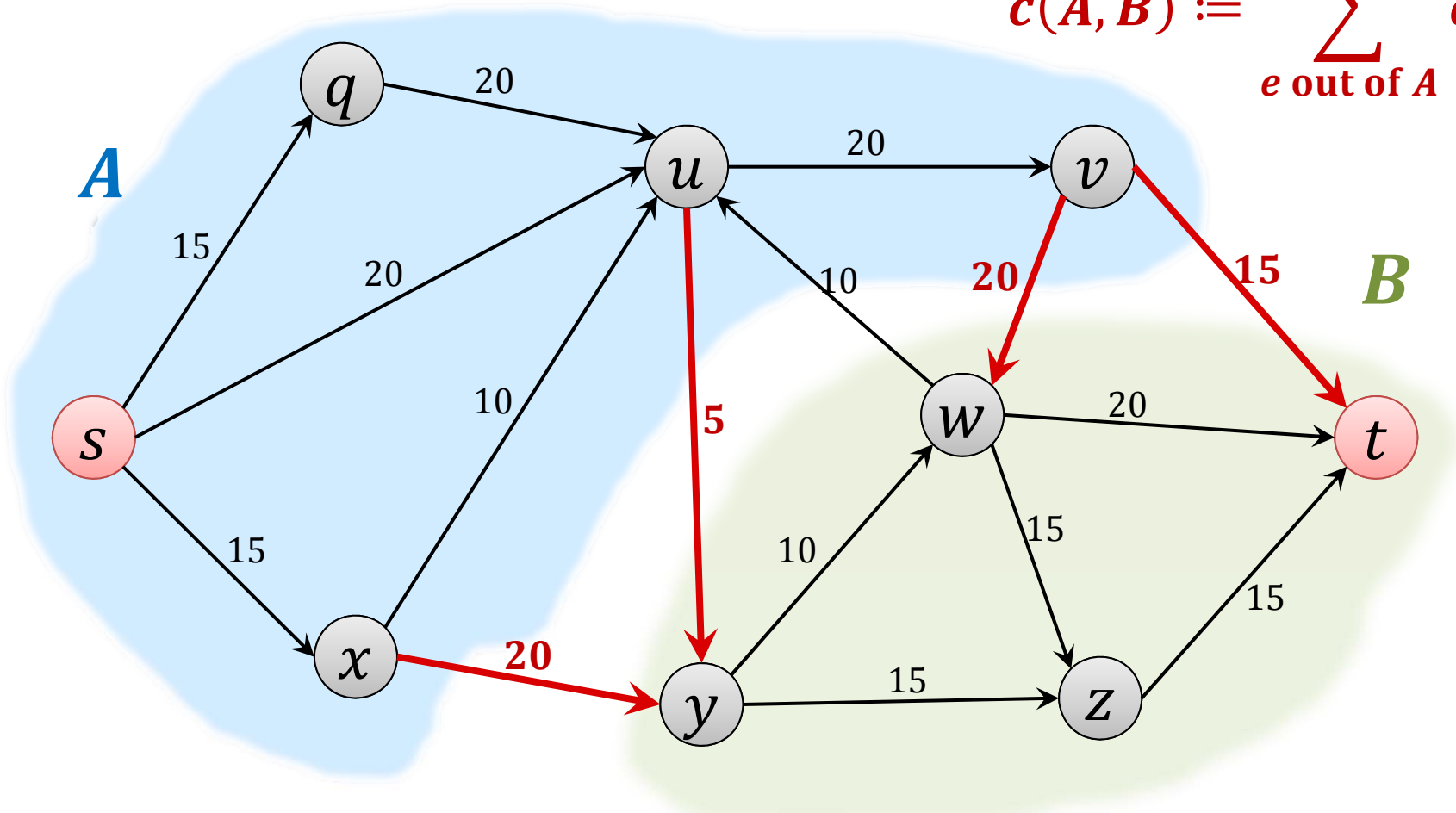


Cut Capacity

Definition:

The **capacity** $c(A, B)$ of an s - t -cut (A, B) is defined as

$$c(A, B) := \sum_{e \text{ out of } A} c_e.$$



Cuts and Flow Value

Lemma: Let f be any s - t flow, and (A, B) any s - t cut. Then,

$$|f| = f^{\text{out}}(A) - f^{\text{in}}(A).$$

Proof:

Cuts and Flow Value

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$$|f| = f^{\text{in}}(B) - f^{\text{out}}(B).$$

Proof:

Upper Bound on Flow Value

Lemma:

Let f be any s - t flow and (A, B) an s - t cut. Then $|f| \leq c(A, B)$.

Proof:

Ford-Fulkerson Gives Optimal Solution



Lemma: If f is an s - t flow such that there is **no augmenting path** in G_f , then there is an s - t cut (A^*, B^*) in G for which

$$|f| = c(A^*, B^*).$$

Proof:

- Define A^* : set of nodes that can be **reached from s** on a path with positive residual capacities **in G_f** :

- For $B^* = V \setminus A^*$, (A^*, B^*) is an s - t cut
 - By definition $s \in A^*$ and $t \notin A^*$