



Chapter 5

Graph Algorithms

Algorithm Theory
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Circulations with Demands

Given: Directed network with positive edge capacities

Sources & Sinks: Instead of one source and one destination, several sources that generate flow and several sinks that absorb flow.

Supply & Demand: sources have supply values, sinks demand values

Goal: Compute a flow such that source supplies and sink demands are exactly satisfied

- The circulation problem is a feasibility rather than a maximization problem

Circulations with Demands: Formally

Given: Directed network $G = (V, E)$ with

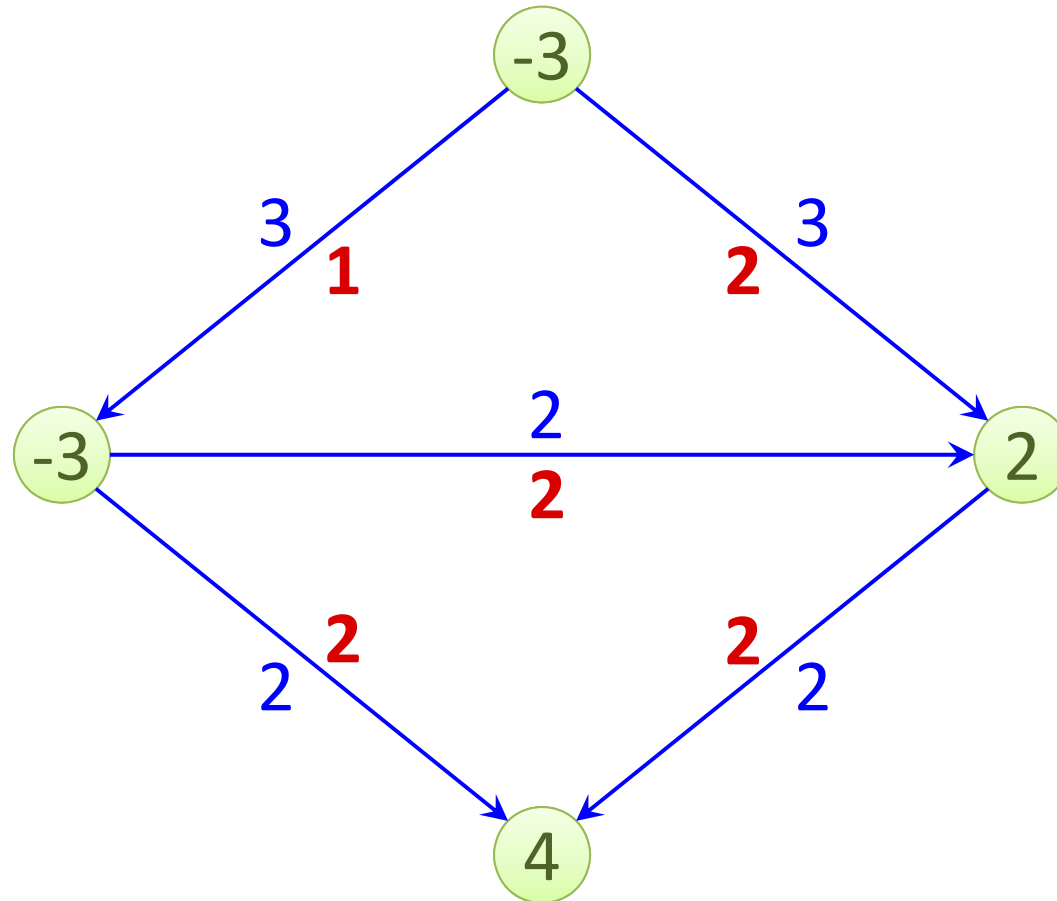
- Edge capacities $c_e > 0$ for all $e \in E$
- Node demands $d_v \in \mathbb{R}$ for all $v \in V$
 - $d_v > 0$: node needs flow and therefore is a sink
 - $d_v < 0$: node has a supply of $-d_v$ and is therefore a source
 - $d_v = 0$: node is neither a source nor a sink

Flow: Function $f: E \rightarrow \mathbb{R}_{\geq 0}$ satisfying

- *Capacity Conditions:* $\forall e \in E: 0 \leq f(e) \leq c_e$
- *Demand Conditions:* $\forall v \in V: f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Objective: Does a flow f satisfying all conditions exist?
If yes, find such a flow f .

Example



Condition on Demands

Claim: If there exists a feasible circulation with demands d_v for $v \in V$, then

$$\sum_{v \in V} d_v = 0.$$

Proof:

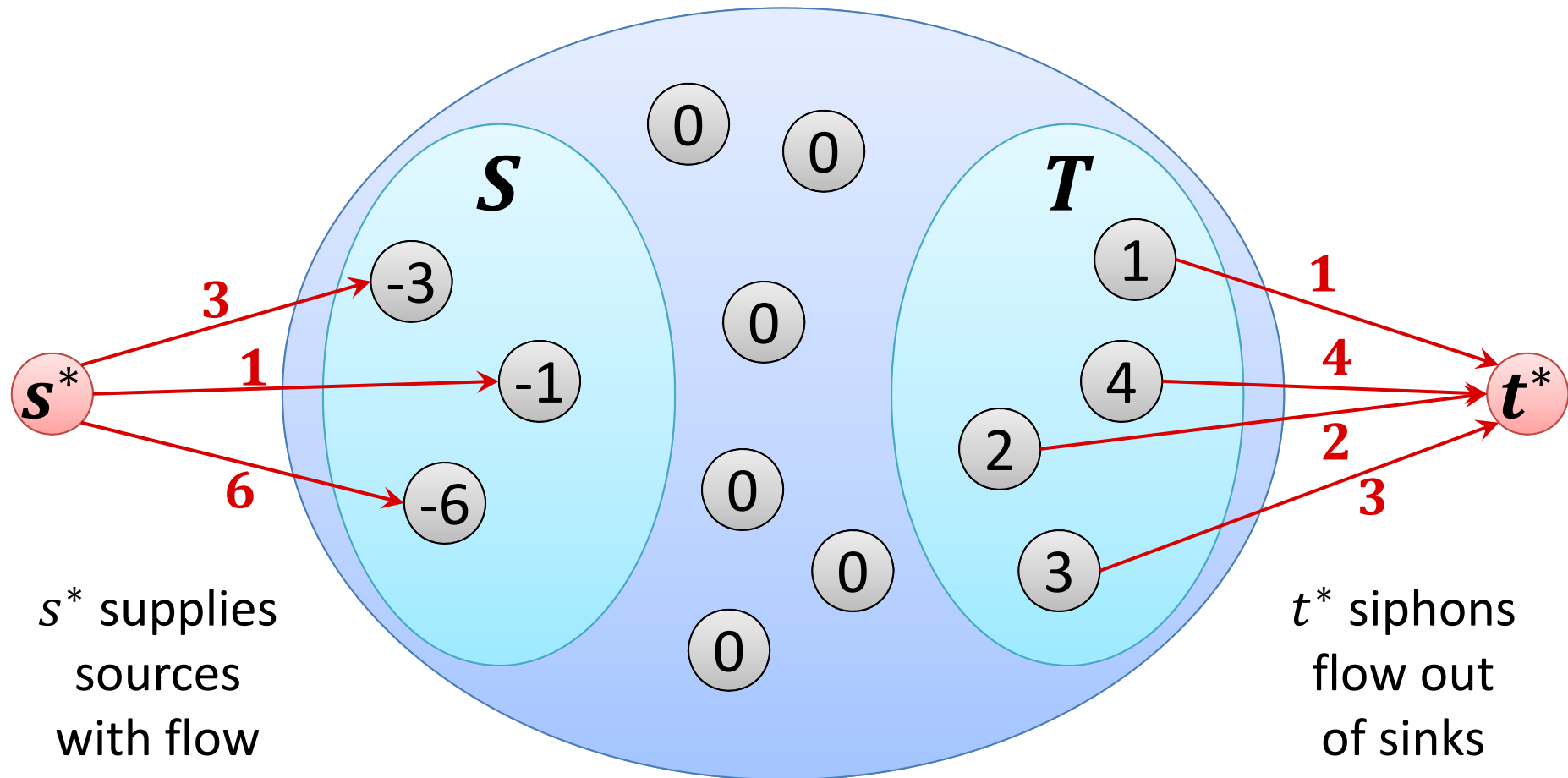
- $\sum_v d_v = \sum_v (f^{\text{in}}(v) - f^{\text{out}}(v))$
- $f(e)$ of each edge e appears twice in the above sum with different signs \rightarrow overall sum is 0

Total supply = total demand:

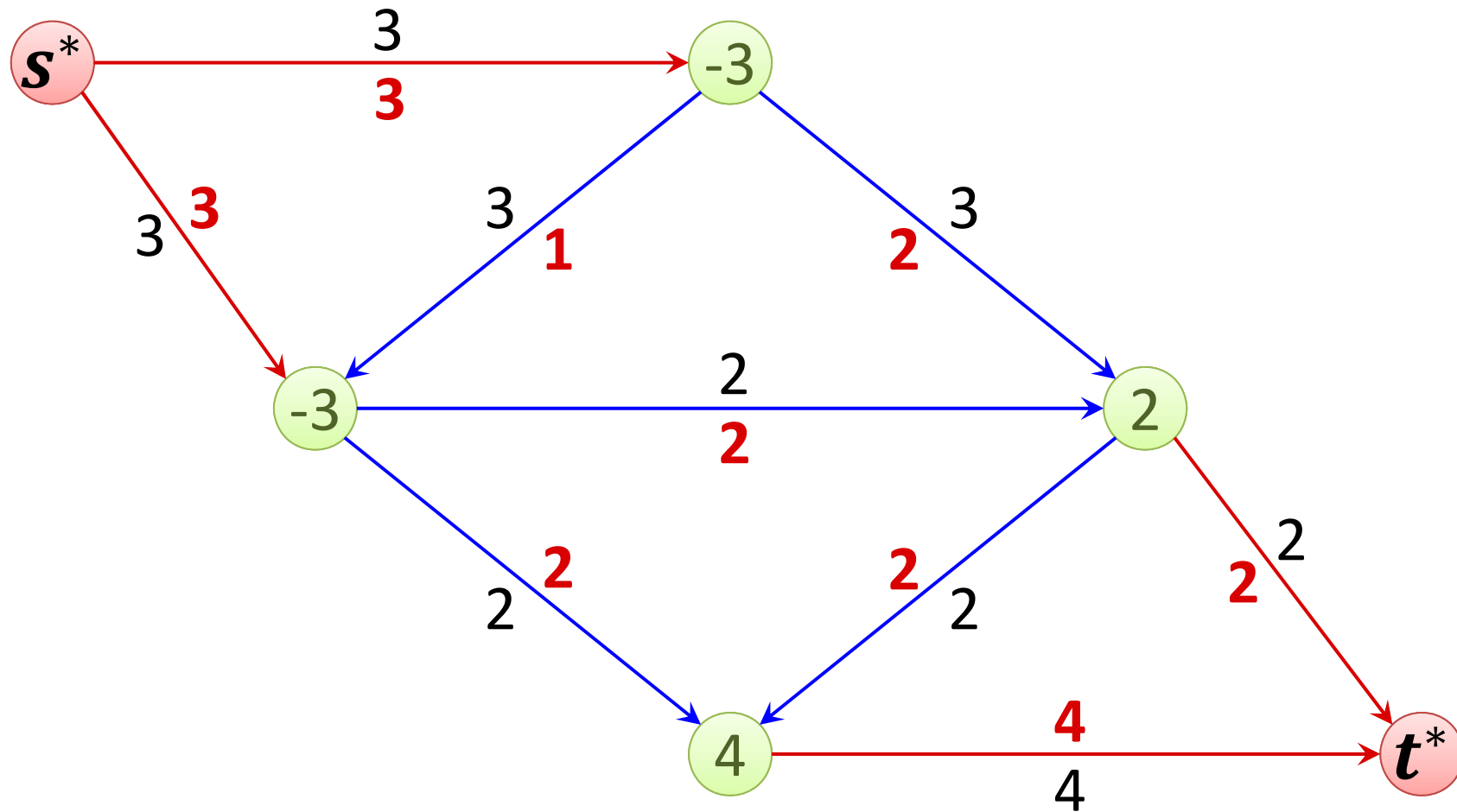
$$\text{Define } D := \sum_{v: d_v > 0} d_v = \sum_{v: d_v < 0} -d_v$$

Reduction to Maximum Flow

- Add “super-source” s^* and “super-sink” t^* to network



Example



Formally...

Reduction: Get graph G' from graph as follows

- Node set of G' is $V \cup \{s^*, t^*\}$
- Edge set is E and edges
 - (s^*, v) for all v with $d_v < 0$, capacity of edge is d_v
 - (v, t^*) for all v with $d_v > 0$, capacity of edge is d_v

Observations:

- Capacity of min s^* - t^* cut is at most D (e.g., the cut $(s^*, V \cup \{t^*\})$)
- A feasible circulation on G can be turned into a feasible flow of value D of G' by saturating all (s^*, v) and (v, t^*) edges.
- Any flow of G' of value D induces a feasible circulation on G
 - (s^*, v) and (v, t^*) edges are saturated
 - By removing these edges, we get exactly the demand constraints

Circulation with Demands

Theorem: There is a feasible circulation with demands $d_v, v \in V$ on graph G if and only if there is a flow of value D on G' .

- If all capacities and demands are integers, there is an integer circulation

The **max flow min cut theorem** also implies the following:

Theorem: The graph G has a feasible circulation with demands $d_v, v \in V$ if and only if for all cuts (A, B) ,

$$\sum_{v \in B} d_v \leq c(A, B).$$

Circulation: Demands and Lower Bounds



Given: Directed network $G = (V, E)$ with

- Edge capacities $c_e > 0$ and **lower bounds $0 \leq \ell_e \leq c_e$ for $e \in E$**
- Node demands $d_v \in \mathbb{R}$ for all $v \in V$
 - $d_v > 0$: node needs flow and therefore is a sink
 - $d_v < 0$: node has a supply of $-d_v$ and is therefore a source
 - $d_v = 0$: node is neither a source nor a sink

Flow: Function $f: E \rightarrow \mathbb{R}_{\geq 0}$ satisfying

- **Capacity Conditions:** $\forall e \in E: \ell_e \leq f(e) \leq c_e$
- **Demand Conditions:** $\forall v \in V: f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Objective: Does a flow f satisfying all conditions exist?
If yes, find such a flow f .

Solution Idea

- Define **initial circulation** $f_0(e) = \ell_e$
Satisfies capacity constraints: $\forall e \in E: \ell_e \leq f_0(e) \leq c_e$

- Define

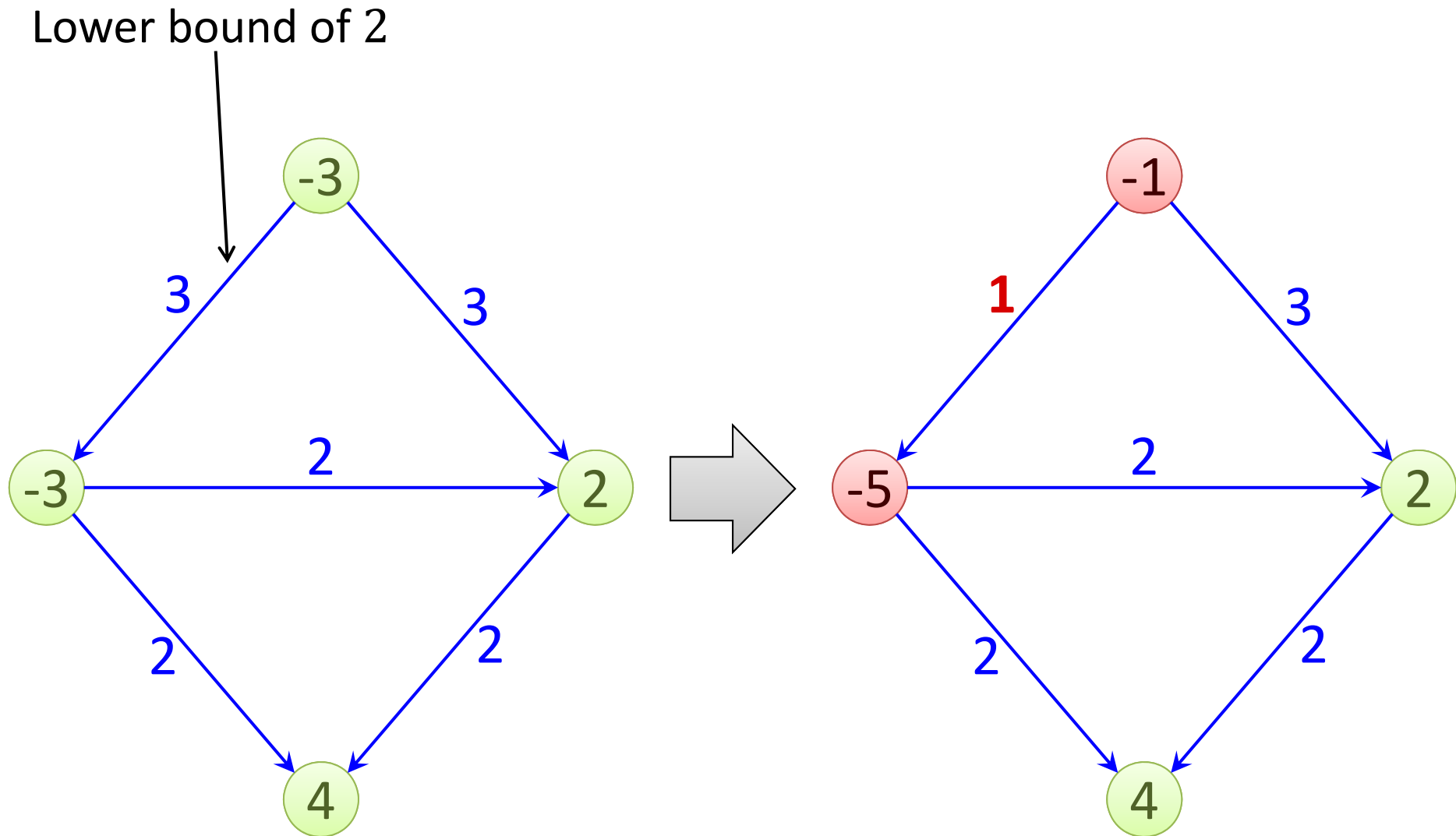
$$L_v := f_0^{\text{in}}(v) - f_0^{\text{out}}(v) = \sum_{e \text{ into } v} \ell_e - \sum_{e \text{ out of } v} \ell_e$$

- If $L_v = d_v$, demand condition is satisfied at v by f_0 , otherwise, we need to superimpose another circulation f_1 such that

$$d'_v := f_1^{\text{in}}(v) - f_1^{\text{out}}(v) = d_v - L_v$$

- Remaining capacity of edge e : $c'_e := c_e - \ell_e$
- We get a circulation problem with new demands d'_v , new capacities c'_e , and **no lower bounds**

Eliminating a Lower Bound: Example



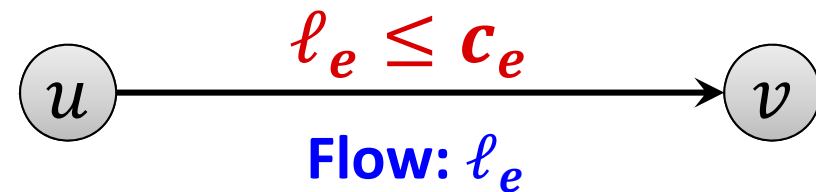
Reduce to Problem Without Lower Bounds



Graph $G = (V, E)$:

- Capacity: For each edge $e \in E: \ell_e \leq f(e) \leq c_e$
- Demand: For each node $v \in V: f^{\text{in}}(v) - f^{\text{out}}(v) = d_v$

Model lower bounds with supplies & demands:



Create Network G' (without lower bounds):

- For each edge $e \in E: c'_e = c_e - \ell_e$
- For each node $v \in V: d'_v = d_v - L_v$

Circulation: Demands and Lower Bounds



Theorem: There is a feasible circulation in G (with lower bounds) if and only if there is feasible circulation in G' (without lower bounds).

- Given circulation f' in G' , $f(e) = f'(e) + \ell_e$ is circulation in G
 - The capacity constraints are satisfied because $f'(e) \leq c_e - \ell_e$
 - Demand conditions:

$$\begin{aligned} f^{\text{in}}(v) - f^{\text{out}}(v) &= \sum_{e \text{ into } v} (\ell_e + f'(e)) - \sum_{e \text{ out of } v} (\ell_e + f'(e)) \\ &= L_v + (d_v - L_v) = d_v \end{aligned}$$

- Given circulation f in G , $f'(e) = f(e) - \ell_e$ is circulation in G'
 - The capacity constraints are satisfied because $\ell_e \leq f(e) \leq c_e$
 - Demand conditions:

$$\begin{aligned} f'^{\text{in}}(v) - f'^{\text{out}}(v) &= \sum_{e \text{ into } v} (f(e) - \ell_e) - \sum_{e \text{ out of } v} (f(e) - \ell_e) \\ &= d_v - L_v \end{aligned}$$

Integrality

Theorem: Consider a circulation problem with integral capacities, flow lower bounds, and node demands. If the problem is feasible, then it also has an integral solution.

Proof:

- Graph G' has only integral capacities and demands
- Thus, the flow network used in the reduction to solve circulation with demands and no lower bounds has only integral capacities
- The theorem now follows because a max flow problem with integral capacities also has an optimal integral solution
- It also follows that with the max flow algorithms we studied, we get an integral feasible circulation solution.

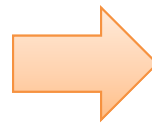
Matrix Rounding

- **Given:** $p \times q$ matrix $D = \{d_{i,j}\}$ of real numbers
- **row i sum:** $a_i = \sum_j d_{i,j}$, **column j sum:** $b_j = \sum_i d_{i,j}$
- **Goal:** **Round** each $d_{i,j}$, as well as a_i and b_j up or down to the next integer so that the sum of rounded elements in each row (column) equals the rounded row (column) sum
- **Original application:** publishing census data

Example:

3.14	6.80	7.30	17.24
9.60	2.40	0.70	12.70
3.60	1.20	6.50	11.30
16.34	10.40	14.50	

original data



3	7	7	17
10	2	1	13
3	1	7	11
16	10	15	

possible rounding

Matrix Rounding

Theorem: For any matrix, there exists a feasible rounding.

Remark: Just rounding to the nearest integer doesn't work

0.35	0.35	0.35	1.05
0.55	0.55	0.55	1.65
0.90	0.90	0.90	

original data

0	0	0	0
1	1	1	3
1	1	1	

rounding to nearest integer

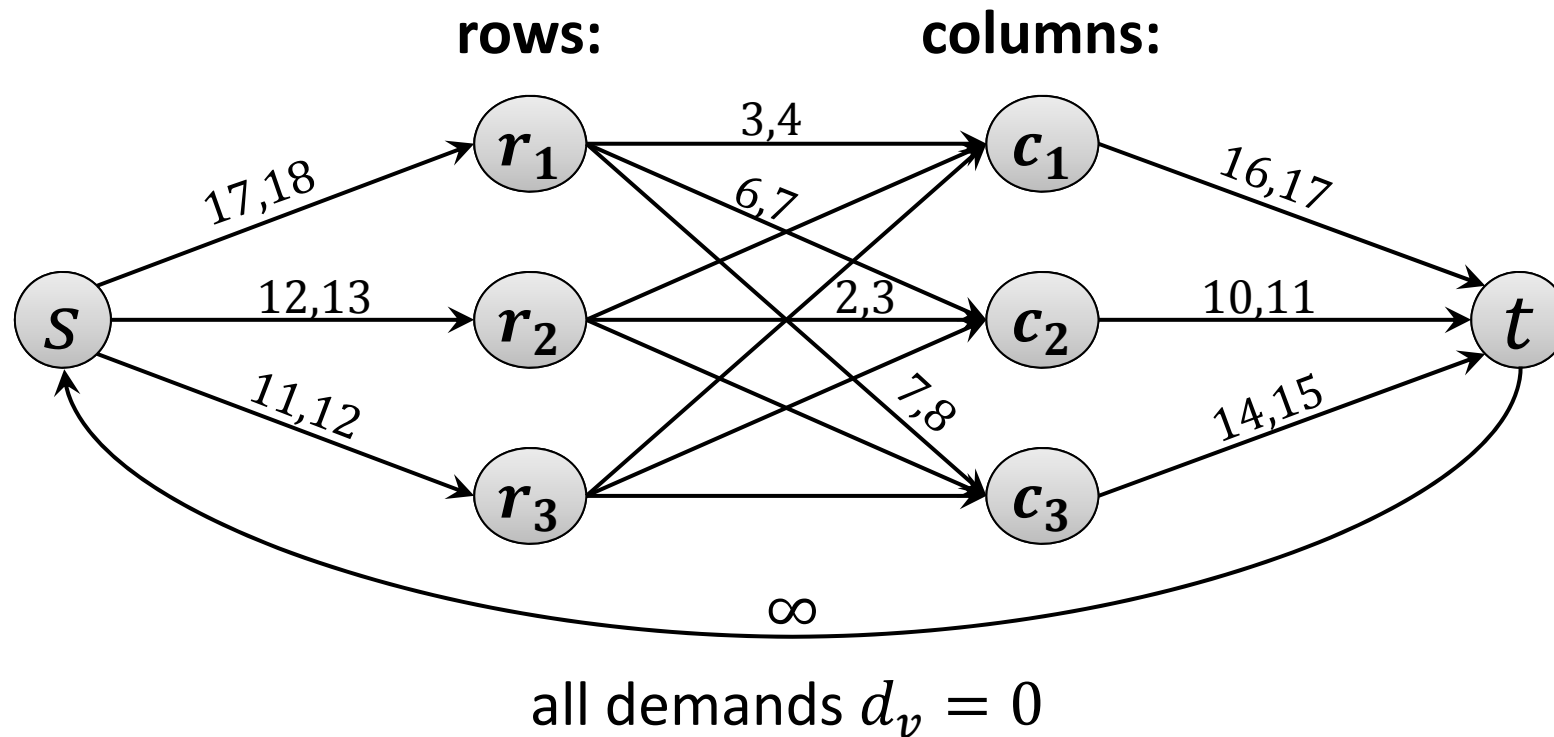
0	0	1	1
1	1	0	2
1	1	1	

feasible rounding

Reduction to Circulation

3.14	6.80	7.30	17.24
9.60	2.40	0.70	12.70
3.60	1.20	6.50	11.30
16.34	10.40	14.50	

Matrix elements and row/column sums give a feasible circulation that satisfies all lower bound, capacity, and demand constraints



Matrix Rounding

Theorem: For any matrix, there exists a feasible rounding.

Proof:

- The matrix entries $d_{i,j}$ and the row and column sums a_i and b_j give a feasible circulation for the constructed network
- Every feasible circulation gives matrix entries with corresponding row and column sums (follows from demand constraints)
- Because all demands, capacities, and flow lower bounds are integral, there is an integral solution to the circulation problem

→ gives a feasible rounding!