Chapter 6
Randomization

Algorithm Theory
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Randomization

Randomized Algorithm:

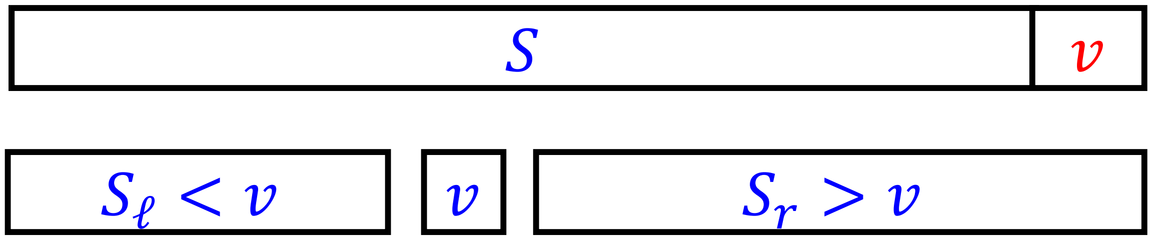
- An algorithm that uses (or can use) random coin flips in order to make decisions

We will see: randomization can be a powerful tool to

- Make algorithms faster
- Make algorithms simpler
- Make the analysis simpler
  - Sometimes it's also the opposite…
- Allow to solve problems (efficiently) that cannot be solved (efficiently) without randomization
  - True in some computational models (e.g., for distributed algorithms)
  - Not clear in the standard sequential model

Randomized Quicksort

Quicksort:



function Quick (S: sequence): sequence;

{returns the sorted sequence S}

begin

  if $\#S \leq 1$ then **return** S

  else { choose pivot element v in S;

    partition S into S_ℓ with elements $< v$,

    and S_r with elements $> v$

    return | Quick(S_ℓ) | v | Quick(S_r) |

end;

Randomized Quicksort Analysis

Randomized Quicksort: pick uniform random element as pivot

Running Time of sorting n elements:

- Let's just count the number of comparisons
- In the partitioning step, all $n-1$ non-pivot elements have to be compared to the pivot
- **Number of comparisons:**

$$\boldsymbol{n-1\ +\ \#\text{comparisons in recursive calls}}$$

- **If rank of pivot is r:**
  recursive calls with $\boldsymbol{r-1}$ and $\boldsymbol{n-r}$ elements

Randomized Quicksort Analysis

Random variables:

- C: total number of comparisons (for a given array of length n)
- R: rank of first pivot
- C_ℓ, C_r: number of comparisons for the 2 recursive calls

$$C = n - 1 + C_\ell + C_r$$

Expectation:

$$\mathbb{E}[C] = \mathbb{E}[n - 1 + C_\ell + C_r]$$

Linearity of Expectation:

$$\mathbb{E}[C] = n - 1 + \mathbb{E}[C_\ell] + \mathbb{E}[C_r]$$

Randomized Quicksort Analysis

Random variables:

- C: total number of comparisons (for a given array of length n)
- R: rank of first pivot
- C_ℓ, C_r: number of comparisons for the 2 recursive calls

$$\mathbb{E}[C] = n - 1 + \mathbb{E}[C_\ell] + \mathbb{E}[C_r]$$

Law of Total Expectation:

$$\mathbb{E}[C] = \sum_{r=1}^{n} \mathbb{P}(R = r) \cdot \mathbb{E}[C|R = r]$$

$$= \sum_{r=1}^{n} \mathbb{P}(R = r) \cdot (n - 1 + \mathbb{E}[C_\ell|R = r] + \mathbb{E}[C_r|R = r])$$

Randomized Quicksort Analysis

We have seen that:

$$\mathbb{E}[C] = \sum_{r=1}^{n} \mathbb{P}(R = r) \cdot (n - 1 + \mathbb{E}[C_\ell | R = r] + \mathbb{E}[C_r | R = r])$$

Define:

- $\boldsymbol{T(n)}$: expected number of comparisons when sorting n elements

$$\mathbb{E}[C] = T(n)$$
$$\mathbb{E}[C_\ell | R = r] = T(r-1)$$
$$\mathbb{E}[C_r | R = r] = T(n-r)$$

Recursion:

$$\boldsymbol{T(n) = \sum_{r=1}^{n} \frac{1}{n} \cdot \big(n - 1 + T(r-1) + T(n-r)\big)}$$
$$\boldsymbol{T(0) = T(1) = 0}$$

Randomized Quicksort Analysis

Theorem: The expected number of comparisons when sorting n elements using randomized quicksort is $T(n) \leq 2n \ln n$.

Proof:

$$T(n) = \sum_{r=1}^{n} \frac{1}{n} \cdot \big(n - 1 + T(r-1) + T(n-r)\big), \qquad T(1) = 0$$

Randomized Quicksort Analysis

Theorem: The expected number of comparisons when sorting n elements using randomized quicksort is $T(n) \leq 2n \ln n$.

Proof:

$$T(n) \leq n - 1 + \frac{4}{n} \cdot \int_1^n x \ln x \, dx$$

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4}$$