



Chapter 9

Parallel Algorithms

Algorithm Theory
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Brent's Theorem

Brent's Theorem: On p processors, a parallel computation can be performed in time

$$T_p \leq \frac{T_1 - T_\infty}{p} + T_\infty.$$

Corollary: Greedy is a 2-approximation algorithm for scheduling.

Corollary: As long as the number of processors $p = O(T_1/T_\infty)$, it is possible to achieve a linear speed-up.

Prefix Sums

- The following works for any associative binary operator \oplus :

associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

All-Prefix-Sums: Given a sequence of n values a_1, \dots, a_n , the all-prefix-sums operation w.r.t. \oplus returns the sequence of prefix sums:

$$s_1, s_2, \dots, s_n = a_1, a_1 \oplus a_2, a_1 \oplus a_2 \oplus a_3, \dots, a_1 \oplus \dots \oplus a_n$$

- Can be computed efficiently in parallel and turns out to be an important building block for designing parallel algorithms

Example: Operator: $+$, input: $a_1, \dots, a_8 = 3, 1, 7, 0, 4, 1, 6, 3$

$$s_1, \dots, s_8 =$$

Computing the Sum

- Let's first look at $s_n = a_1 \oplus a_2 \oplus \dots \oplus a_n$
- Parallelize using a binary tree:

Computing the Sum

Lemma: The sum $s_n = a_1 \oplus a_2 \oplus \dots \oplus a_n$ can be computed in time $O(\log n)$ on an EREW PRAM. The total number of operations (total work) is $O(n)$.

Proof:

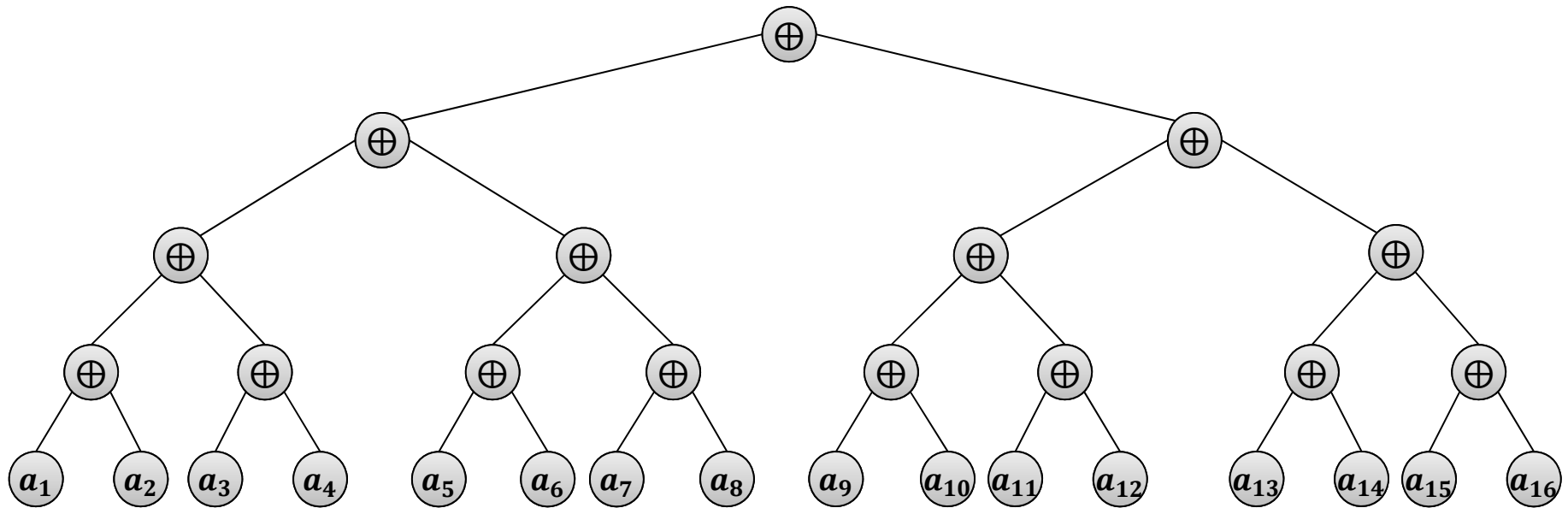
Corollary: The sum s_n can be computed in time $O(\log n)$ using $O(n/\log n)$ processors on an EREW PRAM.

Proof:

- Follows from Brent's theorem ($T_1 = O(n)$, $T_\infty = O(\log n)$)

Getting The Prefix Sums

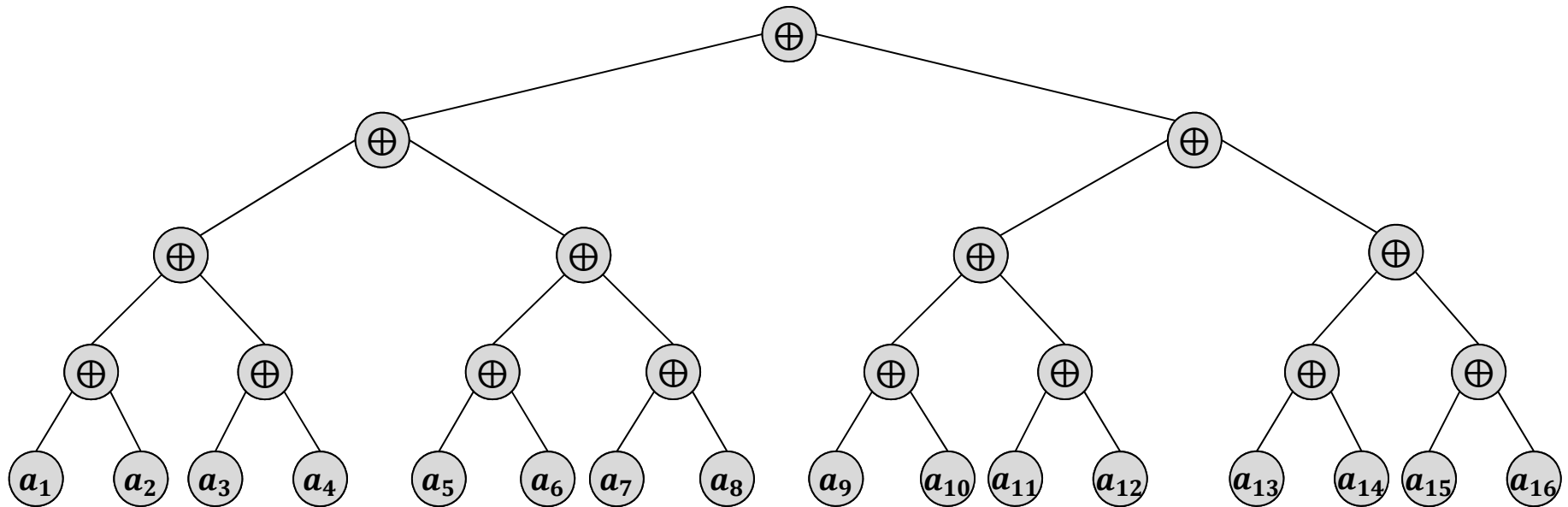
- Instead of computing the sequence s_1, s_2, \dots, s_n let's compute $r_1, \dots, r_n = 0, s_1, s_2, \dots, s_{n-1}$ (0: neutral element w.r.t. \oplus)
 $r_1, \dots, r_n = 0, a_1, a_1 \oplus a_2, \dots, a_1 \oplus \dots \oplus a_{n-1}$
- Together with s_n , this gives all prefix sums
- Prefix sum $r_i = s_{i-1} = a_1 \oplus \dots \oplus a_{i-1}$:



r_{14}
 (s_{13})

Getting The Prefix Sums

Claim: The prefix sum $r_i = a_1 \oplus \dots \oplus a_{i-1}$ is the sum of all the leaves in the left sub-tree of ancestor u of the leaf v containing a_i such that v is in the right sub-tree of u .



Computing The Prefix Sums

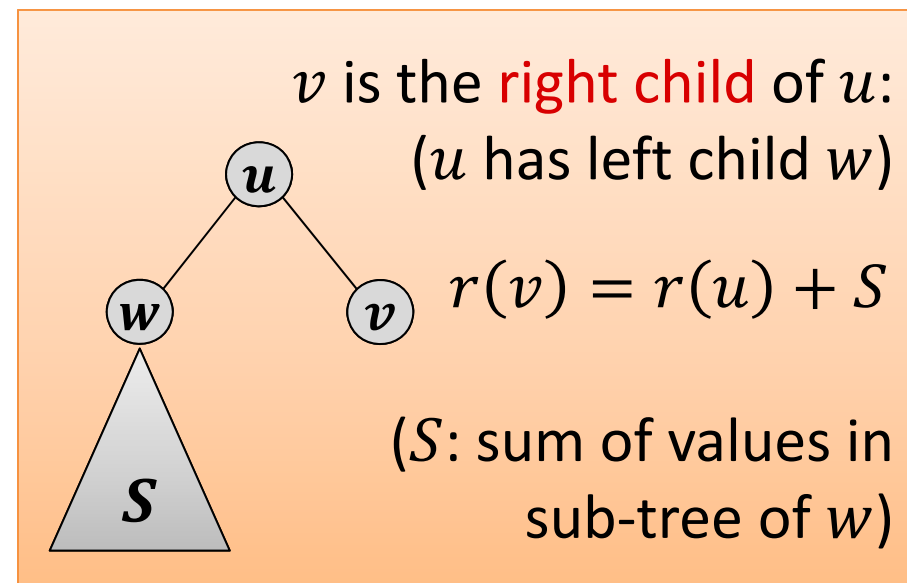
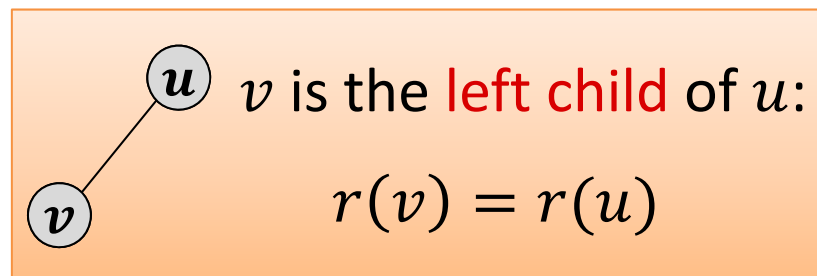
For each node v of the binary tree, define $r(v)$ as follows:

- $r(v)$ is the **sum of the values a_i** at the leaves in all the **left sub-trees of ancestors u** of v such that v is in the right sub-tree of u .

For a **leaf node v** holding value a_i : $r(v) = r_i = s_{i-1}$

For the root node: $r(\text{root}) = 0$

For all **other nodes v** :



Computing The Prefix Sums

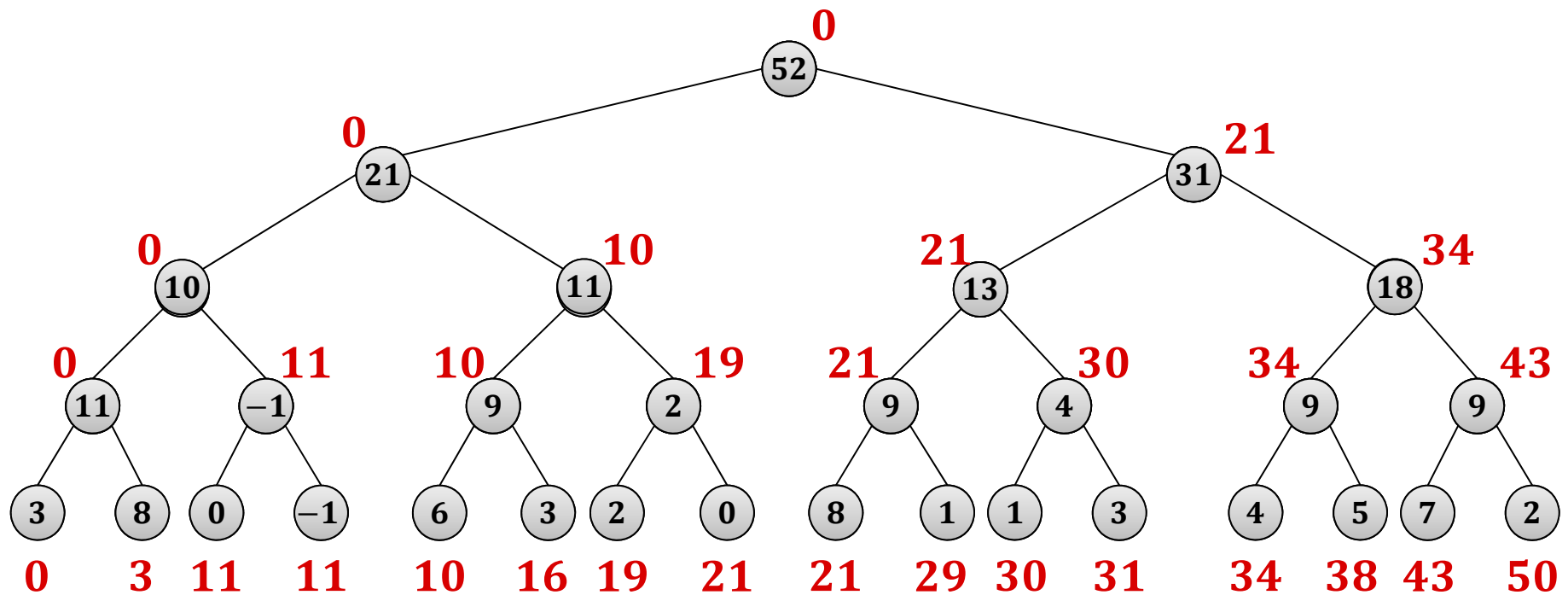
- leaf node v holding value a_i : $r(v) = r_i = s_{i-1}$
- root node: $r(\text{root}) = 0$
- Node v is the left child of u : $r(v) = r(u)$
- Node v is the right child of u : $r(v) = r(u) + S$
 - Where: S = sum of values in left sub-tree of u

Algorithm to compute values $r(v)$:

1. Compute sum of values in each sub-tree (**bottom-up**)
 - Can be done in parallel time $O(\log n)$ with $O(n)$ total work
2. Compute values $r(v)$ **top-down** from root to leaves:
 - To compute the value $r(v)$, only $r(u)$ of the parent u and the sum of the left sibling (if v is a right child) are needed
 - Can be done in parallel time $O(\log n)$ with $O(n)$ total work

Example

1. Compute sums of all sub-trees
 - Bottom-up (level-wise in parallel, starting at the leaves)
2. Compute values $r(v)$
 - Top-down (starting at the root)



Computing Prefix Sums

Theorem: Given a sequence a_1, \dots, a_n of n values, all prefix sums $s_i = a_1 \oplus \dots \oplus a_i$ (for $1 \leq i \leq n$) can be computed in **time $O(\log n)$** using **$O(n/\log n)$ processors** on an EREW PRAM.

Proof:

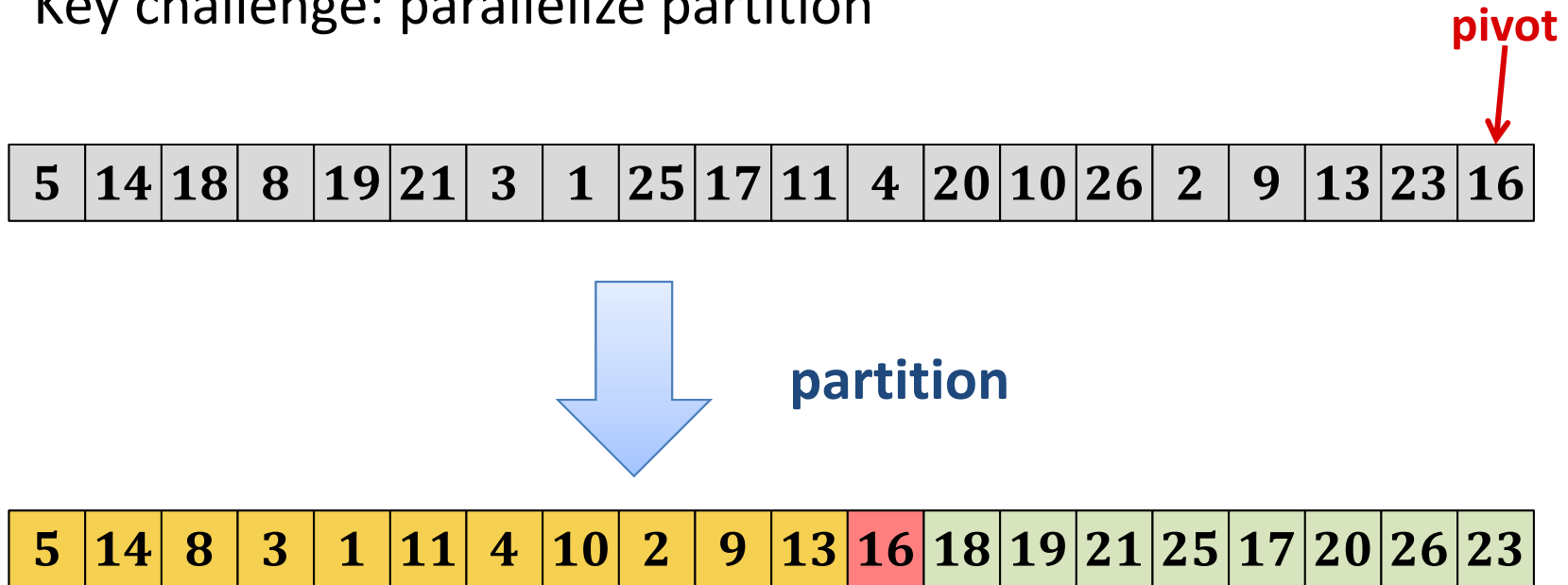
- Computing the sums of all sub-trees can be done in parallel in time $O(\log n)$ using $O(n)$ total operations.
- The same is true for the top-down step to compute the $r(v)$
- The theorem then follows from Brent's theorem:

$$T_1 = O(n), \quad T_\infty = O(\log n) \quad \Rightarrow \quad T_p < T_\infty + \frac{T_1}{p}$$

Remark: This can be adapted to other parallel models and to different ways of storing the value (e.g., array or list)

Parallel Quicksort

- Key challenge: parallelize partition



- How can we do this in parallel?
- For now, let's just care about the values \leq pivot
- What are their new positions

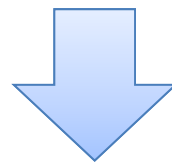
Using Prefix Sums

- Goal: Determine positions of values \leq pivot after partition

5	14	18	8	19	21	3	1	25	17	11	4	20	10	26	2	9	13	23	16
---	----	----	---	----	----	---	---	----	----	----	---	----	----	----	---	---	----	----	----

pivot
↓

1	1	0	1	0	0	1	1	0	0	1	1	0	1	0	1	1	1	0	1
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---



prefix sums

1	2	2	3	3	3	4	5	5	5	6	7	7	8	8	9	10	11	11	12
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	----	----	----	----



partition

5	14	8	3	1	11	4	10	2	9	13	16	18	19	21	25	17	20	26	23
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Partition Using Prefix Sums

- The positions of the entries $>$ pivot can be determined in the same way
- **Prefix sums:** $T_1 = O(n)$, $T_\infty = O(\log n)$
- **Remaining computations:** $T_1 = O(n)$, $T_\infty = O(1)$
- **Overall:** $T_1 = O(n)$, $T_\infty = O(\log n)$

Lemma: The partitioning of quicksort can be carried out in parallel in time $O(\log n)$ using $O\left(\frac{n}{\log n}\right)$ processors.

Proof:

- By Brent's theorem: $T_p \leq \frac{T_1}{p} + T_\infty$

Applying to Quicksort

Theorem: On an EREW PRAM, using p processors, randomized quicksort can be executed in time T_p (in expectation and with high probability), where

$$T_p = O\left(\frac{n \log n}{p} + \log^2 n\right).$$

Proof:

Remark:

- We get optimal (linear) speed-up w.r.t. to the sequential algorithm for all $p = O(n/\log n)$.

Other Applications of Prefix Sums

- Prefix sums are a very powerful primitive to design parallel algorithms.
 - Particularly also by using other operators than +

Example Applications:

- Lexical comparison of strings
- Add multi-precision numbers
- Evaluate polynomials
- Solve recurrences
- Radix sort / quick sort
- Search for regular expressions
- Implement some tree operations
- ...