Albert-Ludwigs-Universität, Inst. für Informatik
Prof. Dr. Fabian Kuhn
M. Ahmadi, A. R. Molla, O. Saukh November 5, 2015

Algorithm Theory, Winter Term 2015/16
Problem Set 3

hand in (hard copy or electronically) by 10:15, Thursday November 12, 2015, tutorial session will be on November 16, 2015

Exercise 1: Greedy Algorithm (6 points)

In the following, a *unit fraction* is a fraction where the numerator is 1 and the denominator is some integer larger than 1. For example 1/4 or 1/384 are unit fractions.

It is well-known that every rational number $0 < q < 1$ can be expressed as a sum of pairwise distinct unit fractions, e.g., we can write $\frac{4}{13}$ as

$$\frac{4}{13} = \frac{1}{5} + \frac{1}{13} + \frac{1}{32} + \frac{1}{65}.$$

Interestingly such a decomposition into distinct unit fractions can be computed using a simple greedy algorithm.

In the following, assume that you are given two positive integers a and b such that $b > a$. Design a greedy algorithm to compute integers $0 < c_1 < c_2 < \cdots < c_k$ such that

$$\frac{a}{b} = \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_k}.$$

Prove that your greedy algorithm always works and that it decomposes $\frac{a}{b}$ into at most a unit fractions.

You can assume that your algorithm can deal with arbitrarily large integer numbers. Note that for the fraction $\frac{4}{13}$, the standard greedy algorithm computes a decomposition which is different from the one given above.

Exercise 2: Matroids (6 points)

We have defined matroids in the lecture. For a matroid (E, I), a maximal independent set $S \in I$ is an independent set that cannot be extended. Thus, for every element $e \in E \setminus S$, the set $S \cup \{e\} \notin I$.

a) Show that all maximal independent sets of a matroid (E, I) have the same size. (This size is called the rank of a matroid.)

b) Consider the following greedy algorithm: The algorithm starts with an empty independent set $S = \emptyset$. Then, in each step the algorithm extends S by the minimum weight element $e \in E \setminus S$ such that $S \cup e \in I$, until S is a maximal independent set. Show that the algorithm computes a maximal independent set of minimum weight.

c) Let E be any finite subset of the natural numbers $\mathbb{N}$ and $k \in \mathbb{N}$ be any natural number. Define a collection of sets $I := \{X \subseteq E : \forall x \neq y \in X,\, x \not\equiv y \pmod{k}\}$. Show that (E, I) is a matroid.