Albert-Ludwigs-Universität, Inst. für Informatik
Prof. Dr. Fabian Kuhn
P. Bamberger, Y. Maus,

Theoretical Computer Science - Bridging Course
Summer Term 2017
Exercise Sheet 11

Hand in (electronically or hard copy) by 12:15 pm, January 29th, 2018

Exercise 1: Understanding FO Logic *(3+2+3 Points)*

Consider the following **first order logical** formulae

$$\begin{aligned}\varphi_1 &:= \forall x R(x,x)\\ \varphi_2 &:= \forall x \forall y\, R(x,y) \to (\exists z R(x,z) \land R(z,y))\\ \varphi_3 &:= \exists x \exists y\, (\neg R(x,y) \land \neg R(y,x))\end{aligned}$$

where x, y are variable symbols and R is a binary predicate. Give an interpretation

(i) I_1 which is a **model** of $\varphi_1 \land \varphi_2$.

(ii) I_2 which is **no model** of $\varphi_1 \land \varphi_2 \land \varphi_3$.

(iii) I_3 which is a **model** of $\varphi_1 \land \varphi_2 \land \varphi_3$.

Exercise 2: Truth Value *(6 Points)*

Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain (or universe) for the variables consists of:

(a) the positive real numbers,

(b) the integers,

(c) the nonzero real numbers.

Exercise 3: Resolution Calculus *(2+4 Points)*

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base KB and formula φ it holds

$$KB \models \varphi \quad \Longleftrightarrow \quad KB \cup \{\neg\varphi\} \models \bot.$$

Remark: $\bot$ is a formula that is unsatisfiable.

Thus, in order to show that KB entails φ, we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus **C** is called *refutation-complete* if for every knowledge base KB

$$KB \models \bot \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \bot.$$

Therefore, if we have a refutation-complete calculus **C**, it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \bot$ in order to prove $KB \models \varphi$.

The *Resolution Calculus*[1] **R** is correct and refutation-complete for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base KB is in CNF if it is of the form $KB = \{C_1, \ldots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \ldots, L_{i,m_i}\}$ each consist of m_i literals $L_{i,j}$
Remark: KB represents the formula $C_1 \wedge \ldots \wedge C_n$ with $C_i = L_{i,1} \vee \ldots \vee L_{i,m_i}$.

The Resolution Calculus has only one inference rule, the *resolution rule*:

$$\mathbf{R}: \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

Remark: L is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in KB (C_1, C_2 may be empty). To show $KB \vdash_{\mathbf{R}} \bot$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses L and $\neg L$. These entail the empty clause (defined as $\Box$), i.e. a contradiction ($\{L, \neg L\} \vdash_{\mathbf{R}} \bot$).
Consider the following propositional formula

$$\psi := (x \wedge y \to z \vee w) \wedge (y \to x) \wedge (z \wedge y \to 0) \wedge (w \wedge y \to 0) \wedge y.$$

Use the **resolution calculus** to show that ψ is unsatisfiable.
Remark: You first have to convert ψ into CNF which you already should have done in one of the previous exercises.
Remark: The 'net' is full of similar exercises. Practice them for the exam!

[1] Complete calculi are unpractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.