University of Freiburg
Department of Computer Science
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Exam Theoretical Computer Science - Bridging Course

Friday, September 07, 2018, 10:00-12:00

Name: ....

Matriculation No.: ....

Signature: ....

Do not open or turn until told so by the supervisor!

- Write your **name** and **matriculation number** on this page and sign the document!
- Write your name on **all sheets**!
- Your **signature** confirms that you feel physically and mentally able to write the exam and that you have answered all questions without any help.
- Write legibly and only use a pen (ink or ball point). Do **not use red**! Do **not use a pencil**!
- This is an **open book exam** therefore printed or hand-written material is allowed.
- However, **no electronic devices** are allowed.
- There are **eight tasks** (with several sub-tasks each) and there is a **total of 100 points**.
- **35 points are sufficient** in order to pass the exam. **70 points** are sufficient to get the best mark.
- Only **one solution per task** is considered! Make sure to strike out alternative solutions, otherwise the one yielding the minimal number of points is considered.
- **Detailed steps** might help you to get more points in case your final result is incorrect.
- The keywords **Show...** or **Prove...** indicate that you need to prove or explain your answer carefully.
- The keywords **Give...** or **State...** indicate that you only need to provide a plain answer.
- You may use information given in a **Hint** without explaining them.
- **Read each task thoroughly** and make sure you understand what is expected from you.
- **Raise your hand** if you have a question regarding the formulation of a task.
- **Use the space below each task and the back of the sheet for your solution.** The last two sheets of this exam are blank and can be used for solutions. If you need additional sheets, raise your hand.

Question	1	2	3	4	5	6	7	8	**Total**
Points									
Maximum	9	16	18	12	10	12	10	13	100

Task 1: Mathematical Proofs (10 Points)

(a) By mathematical induction, prove that for any positive integer n, $6^n - 1$ is divisible by 5. *(5 Points)*

Remark: Integer k is divisible by integer ℓ if there exists an integer s such that $k = s \cdot \ell$.

(b) Prove that every tree of size at least 2 is a bipartite graph. *(5 Points)*

Remark: A graph is bipartite if its set of vertices can be divided into two disjoint and independent sets. An independent set of vertices is a set of vertices in a graph, no two of which are adjacent.

Hint: You can use the fact that every tree T contains a vertex v of degree 1. Moreover, if one removes v from T, the remaining graph is still a tree.

Task 2: DFAs, NFAs, Regular Expressions (11 Points)

Consider the following language $\mathcal{L}$ over alphabet $\mathcal{A} = \{a, b\}$.

Language $\mathcal{L}$ contains all strings in which at least one of the symbols a or b occurs an even number of times. For example, $aba \in \mathcal{L}$ but $ab \notin \mathcal{L}$.

(a) Draw an NFA that recognizes $\mathcal{L}$. *(6 Points)*

(b) Provide a regular expression that recognizes $\mathcal{L}$. *(5 Points)*

Task 3: Context-Free Languages (20 Points)

Consider the following language over alphabet $\{a, b, c\}$.

$L = \{a^*wc^k | w \in \{a, b\}^*,$ and k is the number of a's in $w\}$

(a) Applying the pumping lemma, show that L is not a regular language. *(7 Points)*

(b) Draw a pushdown automata that recognizes L. *(8 Points)*

(c) Give a context-free grammar that recognizes L. *(5 Points)*

Task 4: Regular and Context-Free Languages (14 Points)

(a) Let $L_1, L_2, L_3, \ldots$ be an infinite sequence of regular languages, each of which is defined over a common input alphabet $\Sigma = \{a, b\}$. Let $L = \bigcup_{i=1}^{\infty} L_i$ be the infinite union of $L_1, L_2, L_3, \ldots$.

Is it always the case that L is a regular language? If your answer is YES, give a proof. If your answer is NO, give a counterexample. Explain your answer. *(7 Points)*

(b) Consider alphabet $\Sigma = \{a, b\}$ and arbitrary strings $w_1, w_2, ..., w_{100}$ over Σ. Then, let $L_1, L_2,$ and L_3 be languages defined over Σ, where

- L_1 consists of all possible strings over Σ except the strings $w_1, w_2, ..., w_{100}$.
- L_2 is recognized by an NFA; and
- L_3 is recognized by a PDA.

Prove that $(L_1 \cap L_2) \circ L_3$ is a context-free language.

Hint: First show that L_1 and L_2 are regular. *(7 Points)*

Task 5: Turing machines (10 Points)

Consider alphabet $A = \{1, 2, \ldots, 9\}$. We call a string S over A a *blue* string, if and only if the string consisting of the odd-positioned symbols in S is the reverse of the string consisting of the even-positioned symbols in S. For example $S = 14233241$ is a blue string since the substring of the odd-positioned symbols is 1234 which is the reverse of the substring of the even-positioned symbols, i.e., 4321.

Design a Turing machine which accepts all blue strings over A. You do not need to provide a formal description of the Turing machine but your description has to be detailed enough to explain every possible step of a computation.

Task 6: Undecidability (10 Points)

Fix an enumeration of all Turing machines (that have input alphabet Σ): $\langle M_1 \rangle, \langle M_2 \rangle, \langle M_3 \rangle, \ldots$.

Fix also an enumeration of all words over Σ: $w_1, w_2, w_3, \ldots$.

Prove that language $L = \{w \in \Sigma^* \mid w = w_i, \text{ for some } i, \text{ and } M_i \text{ does not accept } w_i\}$ is not Turing recognizable.

Hint: Try to find a contradiction to the existence of a Turing machine that recognizes L.

Task 7: Complexity Theory (11 Points)

(a) Show that the following language is in class $\mathcal{P}$. *(6 Points)*

9-CYCLE $= \{\langle G \rangle \mid G$ is a graph and contains a cycle of length $9\}$

(b) Is the following statement true or false? Prove your answer. *(5 Points)*

Let L_1 and L_2 be languages in $\mathcal{NP}$, and assume $\mathcal{P} \neq \mathcal{NP}$. Then, if $L_1 \leq_P L_2$ and $L_2 \leq_P L_1$, both L_1 and L_2 are NP-complete.

Task 8: Logic (14 Points)

(a) Consider the following propositional formula

$$\psi := (x \to y \lor z) \land (y \to \neg x) \land (x \land z \to y) \ .$$

(1) Transfer ψ into an equivalent formula in conjunctive normal form (CNF). *(2 Points)*

(2) Use the resolution calculus to show that ψ entails $\neg x$. *(6 Points)*

(b) Consider the following first order logical formulae

$$\begin{aligned}\varphi_1 &:= \forall x\, \neg R(x,x)\\ \varphi_2 &:= \forall x,y,z\ (R(x,y) \land R(y,z) \to R(x,z))\\ \varphi_3 &:= \exists x \forall y\ (x \neq y \to R(x,y))\end{aligned}$$

where x, y are variable symbols and R is a binary predicate. Give an interpretation

(1) I_1 which is a model of $\varphi_1 \land \varphi_2 \land \varphi_3$. *(3 Points)*

(2) I_2 which is a model of $\varphi_1 \land \varphi_2 \land \neg\varphi_3$. *(3 Points)*

***Remark**: No proof required.*