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Theoretical Computer Science - Bridging Course
Winter Term 2018
Exercise Sheet 10

for getting feedback submit (electronically) before the start of the tutorial on 14th of January 2019.

Exercise 1: Propositional Logic: Basic Terms *(2+2+2+2 Points)*

Let $\Sigma := \{p, q, r\}$ be a set of atoms. An interpretation $I : \Sigma \to \{T, F\}$ maps every atom to either true or false. Inductively, an interpretation I can be extended to composite formulae φ over Σ (cf. lecture). We write $I \models \varphi$ if φ evaluates to T (true) under I. In case $I \models \varphi$, I is called a *model* for φ.

For each of the following formulae, give *all* interpretations which are models. Make a truth table and/or use logical equivalencies to find all models (document your steps). Which of these formulae are satisfiable, which are unsatisfiable and which are tautologies?

(a) $\varphi_1 = (p \land \neg q) \lor (\neg p \lor q)$

(b) $\varphi_2 = (\neg p \land (\neg p \lor q)) \leftrightarrow (p \lor \neg q)$

(c) $\varphi_3 = (p \land \neg q) \to \neg(p \land q)$

(d) $\varphi_4 = (p \land q) \to (p \lor r)$

Remark: $a \to b :\equiv \neg a \lor b$, $a \leftrightarrow b :\equiv (a \to b) \land (b \to a)$, $a \not\to b :\equiv \neg(a \to b)$.

Sample Solution

(a) See Table 1. The result shows that φ_1 is a tautology.

(b) See Table 2. The result shows that φ_2 is satisfiable.

(c) φ_3 is equivalent to $\neg(p \land \neg q) \lor (\neg p \lor \neg q)$ which is equivalent to $(\neg p \lor q) \lor (\neg p \lor \neg q)$ which is equivalent to $\neg p \lor q \lor \neg p \lor \neg q$ which is equivalent to $\neg p \lor \neg q \lor q$ which is a tautology as either q or $\neg q$ holds.

(d) See Table 3. The result shows that φ_4 is a tautology.

model	p	q	$p \wedge \neg q$	$\neg p \vee q$	φ_1
✓	0	0	0	1	1
✓	0	1	0	1	1
✓	1	0	1	0	1
✗	1	1	0	1	1

Table 1: Truthtables for Exercises 1 (a).

model	p	q	$\neg p \vee q$	$\neg p \wedge (\neg p \vee q)$	$p \vee \neg q$	φ_2
✓	0	0	1	1	1	1
✗	0	1	1	1	0	0
✗	1	0	0	0	1	0
✗	1	1	1	0	1	0

Table 2: Truthtables for Exercises 1 (b).

model	p	q	r	$p \wedge q$	$p \vee r$	φ_4
✓	0	0	0	0	0	1
✓	0	0	1	0	1	1
✓	0	1	0	0	0	1
✓	0	1	1	0	1	1
✓	1	0	0	0	1	1
✓	1	0	1	0	1	1
✓	1	1	0	1	1	1
✓	1	1	1	1	1	1

Table 3: Truthtables for Exercises 1 (d).

Exercise 2: CNF and DNF *(2+2 Points)*

(a) Convert

$$\psi_1 := (x \land y \to z \lor w) \land (y \to x) \land (z \land y \to 0) \land (w \land y \to 0) \land y$$

into Conjunctive Normal Form (CNF).

(b) Convert

$$\psi_2 := \neg((\neg p \leftrightarrow \neg q) \land (\neg r \to q))$$

into Disjunctive Normal Form (DNF).

Remark: Use the known logical equivalencies given in the lecture slides to do the necessary transformations. State which equivalency you are using in each step. Note that 2a is not ambigious as there are clear rules for interpreting such a formula without additional parentheses.

Sample Solution

(a)

$$(\neg x \lor \neg y \lor z \lor w) \land (\neg y \lor x) \land (\neg z \lor \neg y) \land (\neg w \lor \neg y) \land y \ .$$

(b)

$$(\neg p \land q) \lor (\neg q \land p) \lor (\neg r \land \neg q) \ .$$

Exercise 3: Logical Entailment *(2+2 Points)*

A *knowledge base* KB is a set of formulae over a given set of atoms Σ. An interpretation I of Σ is called a model of KB, if it is a model for *all* formulae in KB. A knowledge base KB *entails* a formula φ (we write $KB \models \varphi$), if *all* models of KB are also models of φ.

Let $KB := \{p \lor (q \land \neg r), \neg r \land p\}$. Show or disprove that KB logically entails the following formulae.

(a) $\varphi_1 := (p \land q) \lor \neg(\neg r \lor p)$

(b) $\varphi_2 := (q \leftrightarrow r) \to p$

Sample Solution

(a) KB does not entail φ_1. Consider the interpretation $I : p \mapsto 1, q \mapsto 0, r \mapsto 0$. Interpretation I is a model for KB but not for φ_1.

(b) Table 4 shows that every model of KB is also a model of φ_2, hence $KB \models \varphi_2$.

Exercise 4: Inference Rules and Calculi *(2+2 Points)*

Let $\varphi_1, \ldots, \varphi_n, \psi$ be propositional formulae. An *inference rule*

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}$$

means that if $\varphi_1, \ldots, \varphi_n$ are 'considered true', then ψ is 'considered true' as well ($n = 0$ is the special case of an axiom). A (propositional) *calculus* **C** is described by a *set* of inference rules.
Given a formula ψ and knowledge base $KB := \{\varphi_1, \ldots, \varphi_n\}$ (where $\varphi_1, \ldots, \varphi_n$ are formulae) we write $KB \vdash_{\mathbf{C}} \psi$ if ψ can be derived from KB by starting from a subset of KB and repeatedly applying inference rules from the calculus **C** to 'generate' new formulae until ψ is obtained.

model of KB	p	q	r	$p \vee (q \wedge \neg r)$	$\neg r \wedge p$	$q \leftrightarrow r$	φ_2	model of φ_2
✗	0	0	0	0	0	1	0	✗
✗	0	0	1	0	0	0	1	✓
✗	0	1	0	1	0	0	1	✓
✗	0	1	1	0	0	1	0	✗
✓	1	0	0	1	1	1	1	✓
✗	1	0	1	1	0	0	1	✓
✓	1	1	0	1	1	0	1	✓
✗	1	1	1	1	0	1	1	✓

Table 4: Truthtable for Exercise 3 (b).

Consider the following two calculi, defined by their inference rules (φ, ψ, χ are arbitrary formulae).

$$\mathbf{C_1}: \quad \frac{\varphi \to \psi, \psi \to \chi}{\varphi \to \chi}, \frac{\neg\varphi \to \psi}{\neg\psi \to \varphi}, \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi, \psi \to \varphi}$$

$$\mathbf{C_2}: \quad \frac{\varphi, \varphi \to \psi}{\psi}, \frac{\varphi \wedge \psi}{\varphi, \psi}, \frac{(\varphi \wedge \psi) \to \chi}{\varphi \to (\psi \to \chi)}$$

Using the respective calculus, show the following derivations (document your steps).

(a) $\{p \leftrightarrow \neg r, \neg q \to r\} \vdash_{\mathbf{C_1}} p \to q$

(b) $\{p \wedge q, p \to r, (q \wedge r) \to s\} \vdash_{\mathbf{C_2}} s$

Remark: Inferences of a given calculus are purely syntactical, i.e. rules only apply in their specific form (much like a grammar) and no other logical transformations not given in the calculus are allowed.

Sample Solution

(a) We use $\mathbf{C_1}$ to derive new formulae until we obtain the desired one.

$$\neg q \to r \overset{\text{2nd rule}}{\vdash_{\mathbf{C_1}}} \neg r \to q$$

$$p \leftrightarrow \neg r \overset{\text{3rd rule}}{\vdash_{\mathbf{C_1}}} p \to \neg r, \neg r \to p$$

$$p \to \neg r, \neg r \to q \overset{\text{1st rule}}{\vdash_{\mathbf{C_1}}} p \to q$$

(b) We use $\mathbf{C_2}$ to derive new formulae until we obtain the desired one.

$$p \wedge q \overset{\text{2nd rule}}{\vdash_{\mathbf{C_2}}} p, q$$

$$p, p \to r \overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} r$$

$$(q \wedge r) \to s \overset{\text{3rd rule}}{\vdash_{\mathbf{C_2}}} q \to (r \to s)$$

$$q, q \to (r \to s) \overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} r \to s$$

$$r, r \to s \overset{\text{1st rule}}{\vdash_{\mathbf{C_2}}} s$$