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Theoretical Computer Science - Bridging Course
Winter Term 2018
Exercise Sheet 11

for getting feedback submit (electronically) before the start of the tutorial on 21th of January 2019.
This is the last exercise sheet!

Exercise 1: Understanding FO Logic *(3+2+3 Points)*

Consider the following **first order logical** formulae

$$\varphi_1 := \forall x R(x,x)$$
$$\varphi_2 := \forall x \forall y\, R(x,y) \to (\exists z R(x,z) \wedge R(z,y))$$
$$\varphi_3 := \exists x \exists y\, (\neg R(x,y) \wedge \neg R(y,x))$$

where x, y are variable symbols and R is a binary predicate. Give an interpretation

(i) I_1 which is a **model** of $\varphi_1 \wedge \varphi_2$.

(ii) I_2 which is **no model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

(iii) I_3 which is a **model** of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

Sample Solution

(i) Pick $I_1 := \langle \mathbb{R}, \cdot^{I_1} \rangle$ where $R^{I_2}(x,y) :\Longleftrightarrow x \leq_{\mathbb{R}} y$.

This is a model because '$\leq_{\mathbb{R}}$' is *reflexive*, therefore fulfills φ_1. Moreover for every $x, y \in \mathbb{R}$ with $x \leq_{\mathbb{R}} y$ we can choose $z := x$, which fulfills $x \leq_{\mathbb{R}} z \wedge z \leq_{\mathbb{R}} y$. Thus φ_2 is also satisfied.

(ii) Pick $I_2 := \langle \mathbb{R}, \cdot^{I} \rangle$ where $R^{I_2}(x,y) = \texttt{false}$.

This is not a model since it violates φ_1, e.g. $R^{I_2}(5,5) = \texttt{false}$.

(iii) Take two disjoint copies of $\mathbb{R}$ and the standard $\leq_{\mathbb{R}}$ relation on each of them; if x and y are from different copies they are not related in $\mathbb{R}$. Formally let

$$I_3 := \langle \{(a,1) \mid a \in \mathbb{R}\} \dot\cup \{(a,2) \mid a \in \mathbb{R}\}, \cdot^{I_3} \rangle$$

where $R^{I_3}((a,g),(b,h)) \Leftrightarrow (g = h \text{ and } a \leq_{\mathbb{R}} b)$.

This is a model because $\leq_{\mathbb{R}}$ is *reflexive*, therefore I_3 fulfills φ_1. Furthermore for every two $x = (a,g)$ and $y = (b,h)$ with $R^{I_3}((a,g),(b,h))$, i.e., $g = h$, we can choose $z := (a,g)$ which fulfills $R^{I_3}((a,g),(a,g)) \wedge R^{I_3}((a,g),(b,h))$. Thus φ_2 is also satisfied. φ_3 is also satisfied, e.g., $(5,1)$ and $(7,2)$ are incomparable, i.e., we have neither $R^{I_3}((5,1),(7,2))$ nor $R^{I_3}((7,2),(5,1))$

Exercise 2: Truth Value *(6 Points)*

Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain (or universe) for the variables consists of:

(a) the positive real numbers,

(b) the integers,

(c) the nonzero real numbers.

Sample Solution

(a) This is false, since no matter how small a positive number x we might choose, if we assume $y = \sqrt{x/2}$, then $x = 2y^2$, and it will not be true that $x \leq y^2$.

(b) This is true, because we can take $x = -1$ as for example.

(c) This is true, since we take $x = -1$.

Exercise 3: Resolution Calculus *(2+4 Points)*

Due to the *Contradiction Theorem* (cf. lecture) for every knowledge base KB and formula φ it holds

$$KB \models \varphi \quad \Longleftrightarrow \quad KB \cup \{\neg\varphi\} \models \bot.$$

Remark: $\bot$ is a formula that is unsatisfiable.

Thus, in order to show that KB entails φ, we show that $KB \cup \{\neg\varphi\}$ entails a contradiction. A calculus **C** is called *refutation-complete* if for every knowledge base KB

$$KB \models \bot \quad \Longrightarrow \quad KB \vdash_{\mathbf{C}} \bot.$$

Therefore, if we have a refutation-complete calculus **C**, it suffices to show $KB \cup \{\neg\varphi\} \vdash_{\mathbf{C}} \bot$ in order to prove $KB \models \varphi$.

The *Resolution Calculus*[1] **R** is correct and refutation-complete for knowledge bases that are given in *Conjunctive Normal Form* (CNF). A knowledge base KB is in CNF if it is of the form $KB = \{C_1, \ldots, C_n\}$ where its clauses $C_i = \{L_{i,1}, \ldots, L_{i,m_i}\}$ each consist of m_i literals $L_{i,j}$
Remark: KB represents the formula $C_1 \wedge \ldots \wedge C_n$ with $C_i = L_{i,1} \vee \ldots \vee L_{i,m_i}$.

The Resolution Calculus has only one inference rule, the *resolution rule*:

$$\mathbf{R}: \quad \frac{C_1 \cup \{L\}, C_2 \cup \{\neg L\}}{C_1 \cup C_2}.$$

Remark: L is a literal and $C_1 \cup \{L\}, C_2 \cup \{\neg L\}$ are clauses in KB (C_1, C_2 may be empty). To show $KB \vdash_{\mathbf{R}} \bot$, you need to apply the resolution rule, until you obtain two conflicting one-literal clauses L and $\neg L$. These entail the empty clause (defined as $\Box$), i.e. a contradiction ($\{L, \neg L\} \vdash_{\mathbf{R}} \bot$).
Consider the following propositional formula

$$\psi := (x \wedge y \rightarrow z \vee w) \wedge (y \rightarrow x) \wedge (z \wedge y \rightarrow 0) \wedge (w \wedge y \rightarrow 0) \wedge y.$$

Use the **resolution calculus** to show that ψ is unsatisfiable.
Remark: You first have to convert ψ into CNF which you already should have done in one of the previous exercises.
Remark: The 'net' is full of similar exercises. Practice them for the exam!

Sample Solution

The formula in CNF is

$$(\neg x \vee \neg y \vee z \vee w) \wedge (\neg y \vee x) \wedge (\neg z \vee \neg y) \wedge (\neg w \vee \neg y) \wedge y.$$

We use the resolution inference rule to derive an unsatisfiable formula

$$\begin{aligned}
\{\neg w, \neg y\}, \{y\} &\vdash_{\mathbf{R}} \{\neg w\} \\
\{\neg z, \neg y\}, \{y\} &\vdash_{\mathbf{R}} \{\neg z\} \\
\{x, \neg y\}, \{y\} &\vdash_{\mathbf{R}} \{x\} \\
\{\neg x, \neg y, z, w\}, \{y\} &\vdash_{\mathbf{R}} \{\neg x, z, w\} \\
\{\neg x, z, w\}, \{\neg w\} &\vdash_{\mathbf{R}} \{\neg x, z\} \\
\{\neg x, z\}, \{\neg z\} &\vdash_{\mathbf{R}} \{\neg x\} \\
\{\neg x\}, \{x\} &\vdash_{\mathbf{R}} \Box
\end{aligned}$$

[1] Complete calculi are unpractical, since they have too many inference rules. More inference rules make automated proving with a computer significantly more complex. The Resolution Calculus is an appropriate technique to avoid this additional complexity, since it has only one inference rule.