Algorithms and Data Structures

Linked Lists, Binary Search Trees

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Structure

Structure:

- We have a set of keys mapped to values
- We have an ordering $<$ applied to the keys
- We need the following operations:
  - `insert(key, value)`: insert the given pair
  - `remove(key)`: remove the pair with the given key
  - `lookup(key)`: find the element with the given key, if it is not available find the element with the next smallest key
  - `next()/previous()`: returns the element with the next bigger/smaller key. This enables iteration over all elements

Application examples:

- Example: database for books, products or apartments
- Large number of records (data sets / tuples)
- Typical query: return all apartments with a monthly rent between 400€ and 600€
  - This is called a range query
  - We can implement this with a combination of `lookup(key)` and `next()`
  - It's not essential that an apartment exists with **exactly** 400€ monthly rent
- We do not want to sort all elements every time on an insert operation
- How could we implement this?

Static array:

3	5	9	14	18	21	26	40	41	42	43	46

- `lookup` in time $O(\log n)$
  - With **binary search**
  - Example: `lookup(41)`
- `next` / `previous` in time $O(1)$
  - They are next to each other
- `insert` and `remove` up to $\Theta(n)$
  - We have to copy up to n elements

Hash map:

- `insert` and `remove` in $O(1)$

  If the hash table is big enough and we use a good hash function

- `lookup` in time $O(1)$

  If element with **exactly** this key exists, otherwise we get None as result

- `next` / `previous` in time up to $\Theta(n)$

  Order of the elements is independent of the order of the keys

Linked list:

- Runtimes for doubly linked lists:
  - `next` / `previous` in time $O(1)$
  - `insert` and `remove` in $O(1)$
  - `lookup` in time $\Theta(n)$
- Not yet what we want, but structure is related to binary search trees
- Let's have a closer look

Linked list:

- Dynamic datastructure
- Number of elements changeable
- Data elements can be simple types or composed data structures
- Elements are linked through references / pointer to the predecessor / successor
- Single / doubly linked lists possible

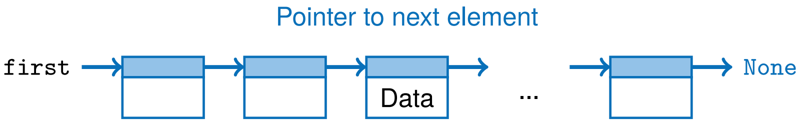

Figure: Linked list

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Properties in comparison to an array:

- Minimal extra space for storing pointer
- We do not need to copy elements on `insert` or `remove`
- The number of elements can be simply modified
- No direct access of elements
  $\Rightarrow$ We have to iterate over the list

List with head / last element pointer:

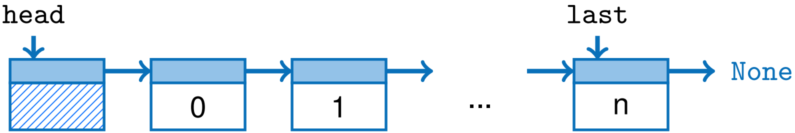

Figure: Singly linked list

- Head element has pointer to first list element
- May also hold additional information:
  - Number of elements

Doubly linked list:

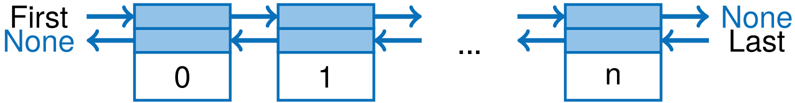

Figure: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element
- Iterate forward and backward

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode=None):
        self.value = value
        self.nextNode = nextNode
```

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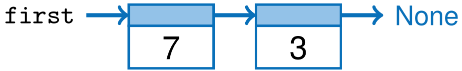
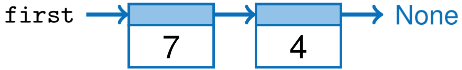
Creating linked lists - Python:

- `first = Node(7)`

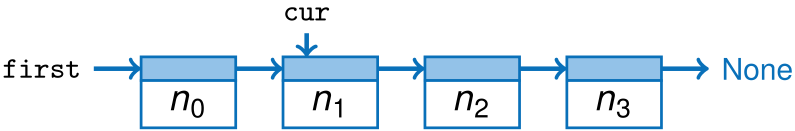
- `first.nextNode = Node(3)`



- `first.nextNode.value = 4`

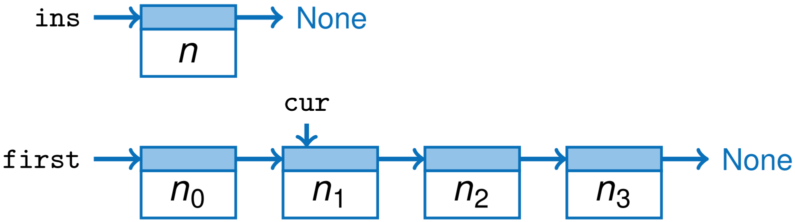


Inserting a node after node `cur`:



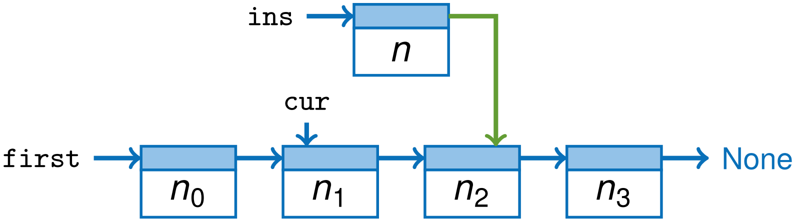
Inserting a node after node `cur`:

- `ins = Node(n)`



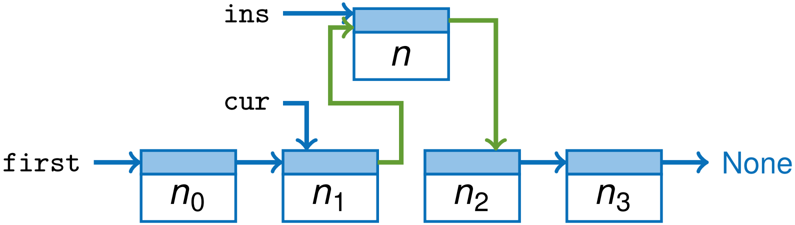
Inserting a node after node `cur`:

- `ins.nextNode = cur.nextNode`



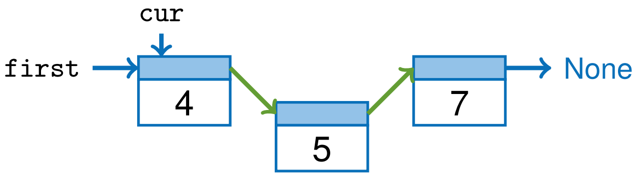
Inserting a node after node `cur`:

- `cur.nextNode = ins`

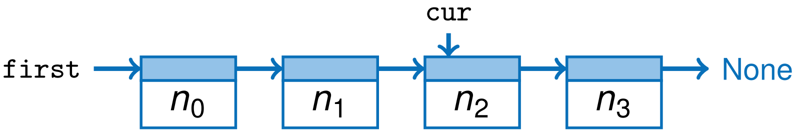


Inserting a node after node `cur` - single line of code:

- `cur.nextNode = Node(value, cur.nextNode)`



Removing a node `cur`:

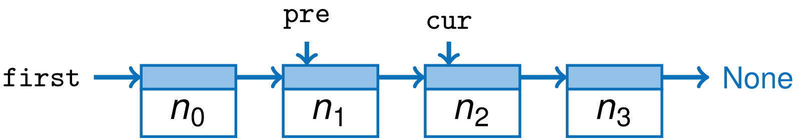


Removing a node `cur`:

- Find the predecessor of `cur`:

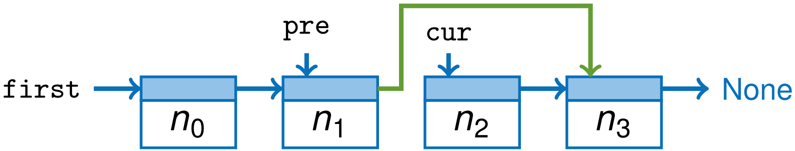
```
pre = first
while pre.nextNode != cur:
    pre = pre.nextNode
```

- Runtime of $O(n)$
- Does not work for first node!

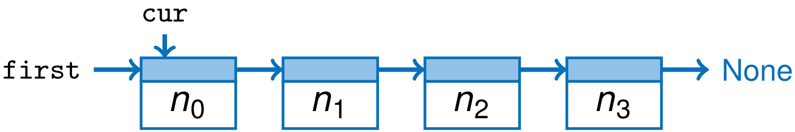


Removing a node `cur`:

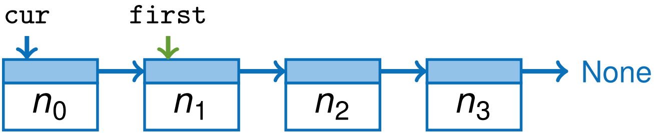
- Update the pointer to the next element:
  `pre.nextNode = cur.nextNode`
- `cur` will get destroyed automatically if no more references exist (`cur=None`)



Removing the first node:



- Update the pointer to the next element:
  `first = first.nextNode`
- `cur` will get automaticly destroyed if no more references exist (`cur=None`)



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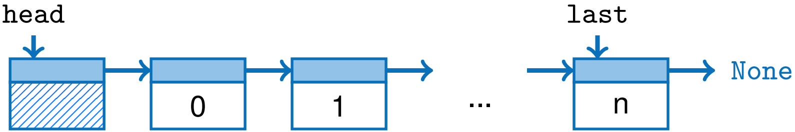
Removing a node `cur`: (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```

Using a `head` node:

- Advantage:
  - Deleting the first node is no special case
- Disadvantage
  - We have to consider the first node at other operations
  - Iterating all nodes
  - Counting of all nodes
  - ...



```
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```

Linked Lists

Implementation - LinkedList - Python

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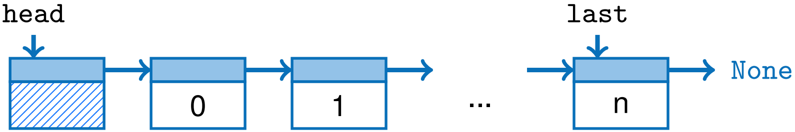
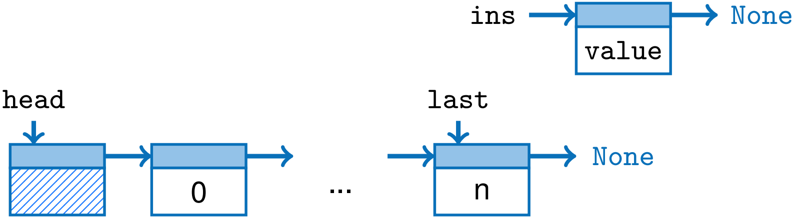
```
def append(self, value):
    ...

def insertAfter(self, cur, value):
    ...

def remove(self, cur):
    ...

def get(self, position):
    ...

def contains(self, value):
    ...
```

Head, last:



- `Head` points to the first node, `last` to the last node
- We can append elements to the end of the list in $O(1)$ through the `last` node
- We have to keep the pointer to `last` updated after all operations

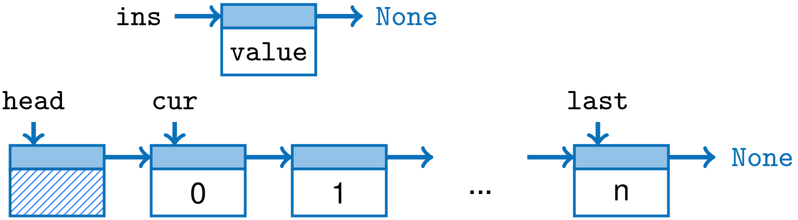
Appending an element:




```python
def append(self, value):
    last.nextNode = Node(value)
    last = last.NextNode
    itemCount += 1
```

- The pointer to `last` avoids the iteration of the whole list

Inserting after node `cur`:

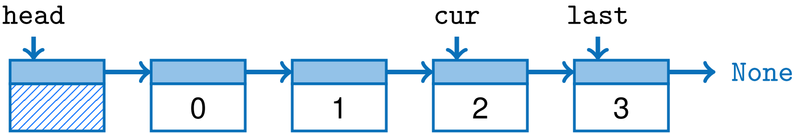


Inserting after node `cur`:

- The pointer to `head` is not modified

```
def insertAfter(self, cur, value):
    if cur == last:
        # also update last node
        append(value)
    else:
        # last node is not modified
        cur.nextNode = Node(value, \
            cur.nextNode)
        itemCount += 1
```

Remove node `cur`:



Remove node `cur`:

- Searching the predecessor in $O(n)$

```
def remove(self, cur):
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
    itemCount -= 1

    if pre.nextNode == None:
        last = pre
```

Getting a reference to node at `pos`:

- Iterate the entries of the list until position in $O(n)$

```
def get(self, pos):
    if pos < 0 or pos >= itemCount:
        return None

    cur = head
    for i in range(0, pos):
        cur = cur.nextNode

    return cur
```

Searching a `value`:

- First element is head without an assigned value
- Iterate the entries of the list until value found in $O(n)$

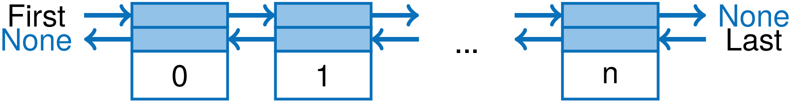
```
def contains(self, value):
    cur = head

    for i in range(0, itemCount):
        cur = cur.nextNode
        if cur.value == value:
            return True

    return False
```

Runtime:

- Singly linked list:
  - `next` in $O(1)$
  - `previous` in $\Theta(n)$
  - `insert` in $O(1)$
  - `remove` in $\Theta(n)$
  - `lookup` in $\Theta(n)$
- Better with doubly linked lists

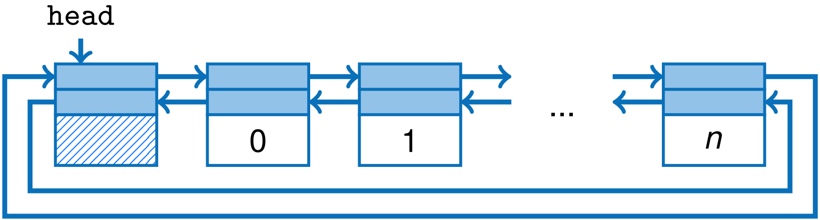
Doubly linked list:

- Each node has a reference to its successor and its predecessor
- We can iterate the list forward and backward



Doubly linked list:

- It is helpful to have a head node
- We only need one head node if we cyclically connect the list



Runtime of doubly linked list:

- `next` and `previous` in $O(1)$

  Each element has a pointer to pred-/sucessor

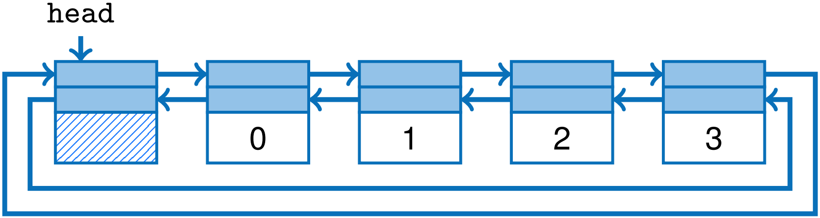
- `insert` and `remove` in $O(1)$

  A constant number of pointers needs to be modified

- `lookup` in $\Theta(n)$

  Even if the elements are sorted we can only retrieve them in $\Theta(n)$ Why?

Linked list in book:



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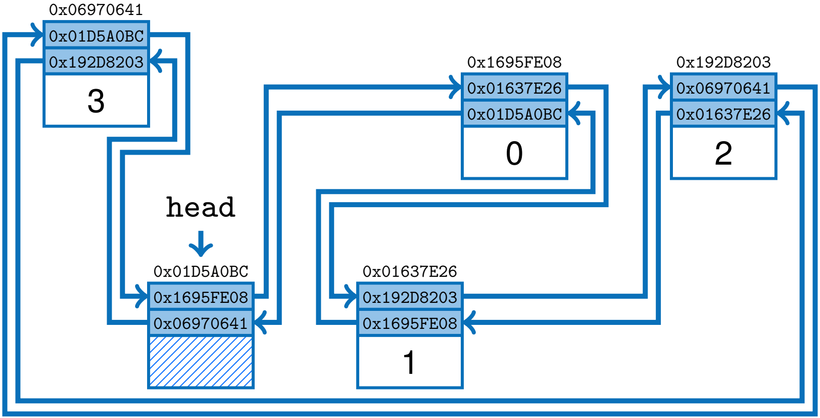
Linked list in memory:



Runtime of a search tree:

- `next` and `previous` in $O(1)$

  Pointers corresponding to linked list

- `insert` and `remove` in $O(\log n)$

- `lookup` in $O(\log n)$

  The structure helps searching efficiently

Idea:

- We define a total order for the search tree
- All nodes of the left subtree have smaller keys than the current node
- All nodes of the right subtree have bigger keys than the current node

- Edge direction indicates ordering

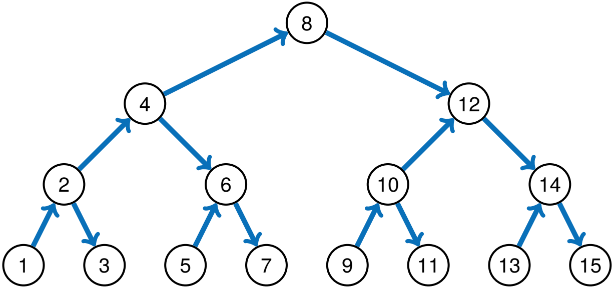


Figure: a binary search tree

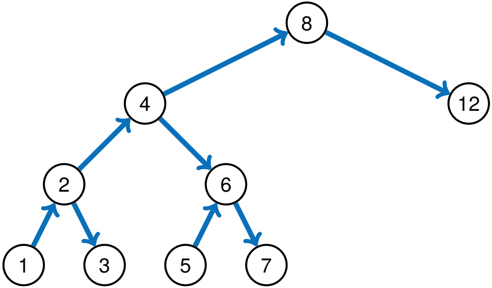


Figure: another binary search tree

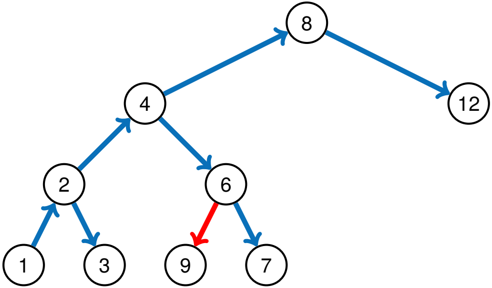
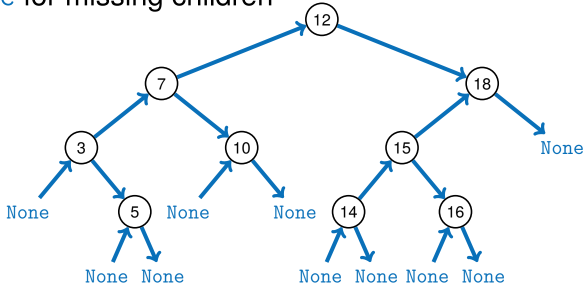


Figure: **not** a binary search tree

Implementation:

- For the heap we had all elements stored in an array
- Here we link all nodes through pointers / references, like linked lists
- Each node has a pointer / reference to its children (`leftChild` / `rightChild`)
- `None` for missing children



Implementation:

- We create a sorted doubly linked list of all elements
- This enables an efficient implementation of (`next` / `previous`)

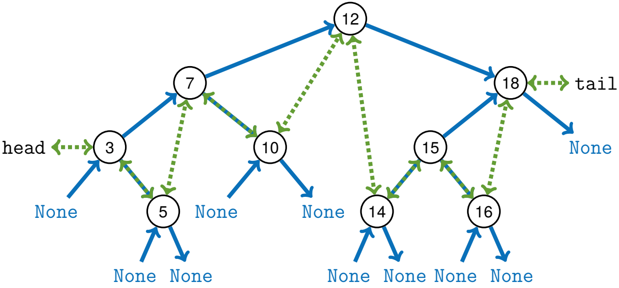


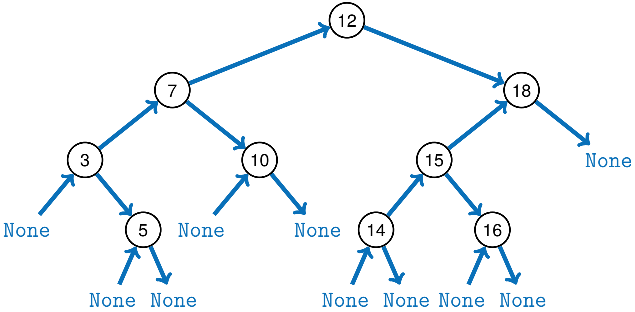
Figure: binary search tree with links

Lookup:

- Definition:
  " Search the element with the given key. If no element is found return the element with the next (bigger) key. "
- We search from the root downwards:
  - Compare the searched key with the key of the node
  - Go to the left / right until the child is `None` or the key is found
  - If the key is not found return the next bigger one

For each node applies the total order:

keys of left subtree < `node.key` < keys of right subtree



Examples:

`lookup(14)`

`lookup(6)`

`lookup(19)`

Figure: binary search tree with total order "<"

Insert:

- We search for the key in our search tree
- If a node is found we replace the value with the new one
- Else we insert a new node
- If the key was not present we get a `None` entry
- We insert the node there

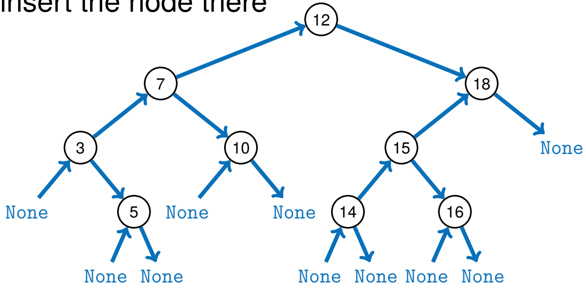


Figure: Binary search tree with total order “<”

Remove: case 1: the node “5” has no children

- Find parent of node “5” (“6”)
- Set left / right child of node “6” to `None` depending on position of node “5”

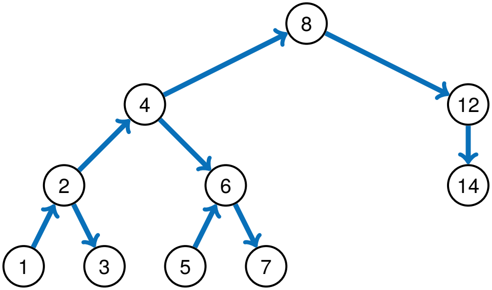


Figure: Binary search tree with total order “<”

Remove: Case 1: The node “5” has no children

- Find parent of node “5” (“6”)
- Set left / right child of node “6” to `None` depending on position of node “5”

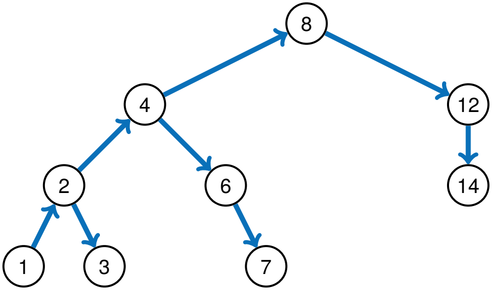


Figure: binary search tree after deleting node “5”

Remove: Case 2: The node “12” has one child

- Find the child of node “12” (“14”)
- Find the parent of node “12” (“8”)
- Set left / right child of node “8” to “14” depending on position of node “12” (skip node “14”)

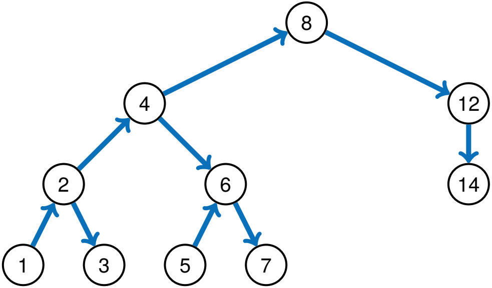


Figure: binary search tree with total order “<”

Remove: Case 2: The node “12” has one child

- Find the child of node “12” (“14”)
- Find the parent of node “12” (“8”)
- Set left / right child of node “8” to “14” depending on position of node “12” (skip node “14”)

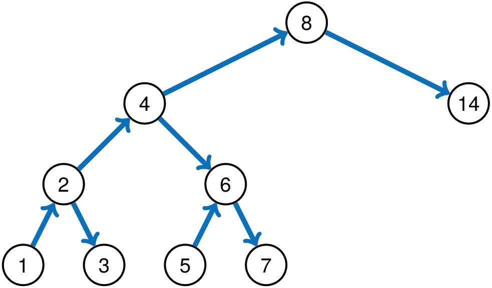
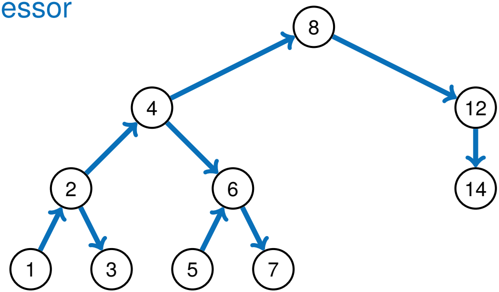


Figure: binary search tree after delting node “12”

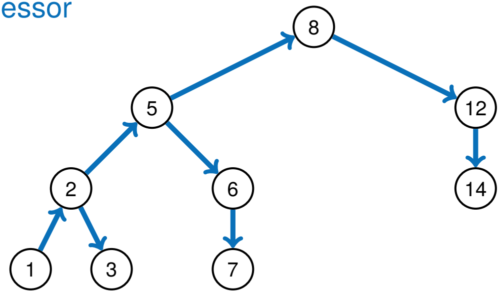
Remove: Case 3: The node “4” has two children

- Find the successor of node “4” (“5”)
- Replace the value of node “4” with the value of node “5”
- Delete node “5” (the successor of node “4”) with remove-case 1 or 2
- There is no left node because we are deleting the predecessor



Remove: Case 3: The node “4” has two children

- Find the successor of node “4” (“5”)
- Replace the value of node “4” with the value of node “5”
- Delete node “5” (the successor of node “4”) with remove-case 1 or 2
- There is no left node because we are deleting the predecessor



How long takes `insert` and `lookup`?

- Up to $\Theta(d)$, with d being the depth of the tree (The longest path from the root to a leaf)
- Best case with $d = \log n$ the runtime is $\Theta(\log n)$
- Worst case with $d = n$ the runtime is $\Theta(n)$
- If we **always** want to have a runtime of $\Theta(\log n)$ then we have to rebalance the tree

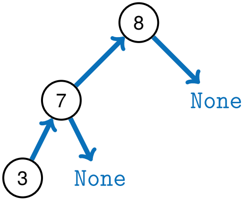


Figure: degenerated binary tree $d = n$

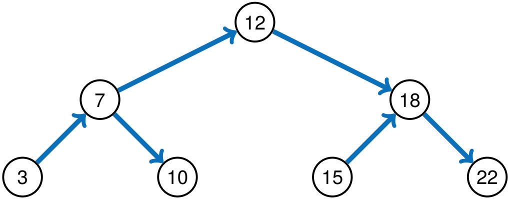


Figure: complete binary tree $d = \log n$

- **Course literature**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

- **Linked List**

[Wik] Linked list
https://en.wikipedia.org/wiki/Linked_list

- **Binary Search Tree**

[Wik] Binary search tree
https://en.wikipedia.org/wiki/Binary_search_tree