# Algorithms and Data Structures

## Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science

Algorithms and Data Structures, January 2019

# Structure

**Binary search tree:**

- With `BinarySearchTree` we could perform an `lookup` or `insert` in $O(d)$, with $d$ being the depth of the tree
- Best case: $d \in O(\log n)$, keys are inserted randomly
- Worst case: $d \in O(n)$, keys are inserted in ascending / descending order $(20, 19, 18, \ldots)$

**Gnarley trees:**

- http://people.ksp.sk/~kuko/bak

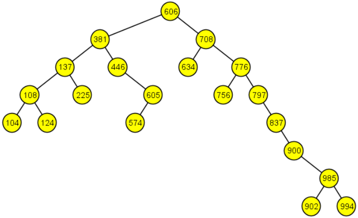



Figure: Binary search tree with random insert [Gna]

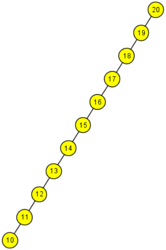



Figure: Binary search tree with descending insert [Gna]

**Balanced trees:**

- We do not want to rely on certain properties of our key set
- We explicitly want a depth of $O(\log n)$
- We rebalance the tree from time to time

UNI FREIBURG

**How do we get a depth of $O(\log n)$?**

- **AVL-Tree:**
  - Binary tree with 2 children per node
  - Balancing via “rotation”
- **(a,b)-Tree** or **B-Tree:**
  - Node has between *a* and *b* children
  - Balancing through splitting and merging nodes
  - Used in databases and file systems
- **Red-Black-Tree:**
  - Binary tree with “black” and “red” nodes
  - Balancing through “rotation” and “recoloring”
  - Can be interpreted as (2, 4)-tree
  - Used in C++ `std::map` and Java `SortedMap`

**AVL-Tree:**

- Gregory Maximovich **A**delson-**V**elskii, Yevgeniy Mikhailovlovich **L**andis (1963)
- Search tree with modified `insert` and `remove` operations while satisfying a depth condition
- Prevents degeneration of the search tree
- Height difference of left and right subtree is at maximum one
- With that the height of the search tree is always $O(\log n)$
- We can perform all basic operations in $O(\log n)$

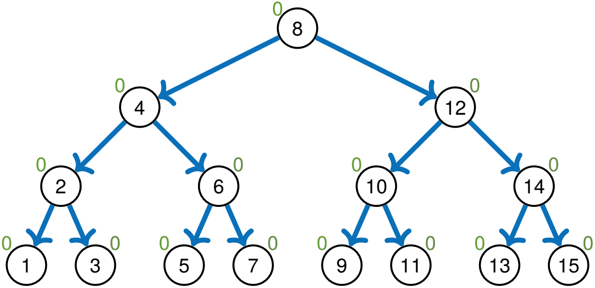



Figure: Example of an AVL-Tree

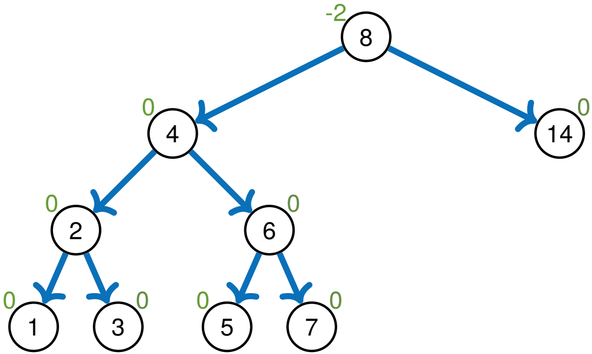



Figure: **Not** an AVL-Tree

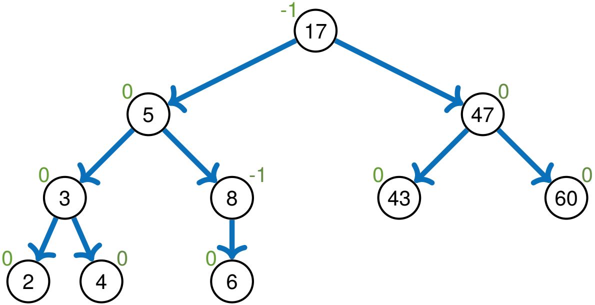



Figure: Another example of an AVL-Tree

**Rotation:**

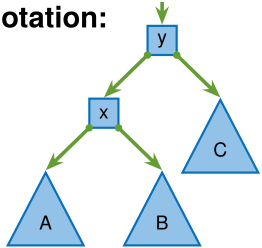



Figure: Before rotating

$\Rightarrow$

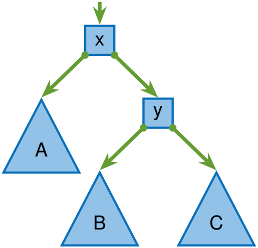



Figure: After rotating

- Central operation of rebalancing
- After rotation to the right:
  - Subtree $A$ is a layer higher and subtree $C$ a layer lower
  - The parent child relations between nodes $x$ and $y$ have been swapped

**AVL-Tree:**

- If a height difference of $\pm 2$ occurs on an `insert` or `remove` operation the tree is rebalanced
- Many different cases of rebalancing
- **Example:** `insert` of $1, 2, 3, \ldots$

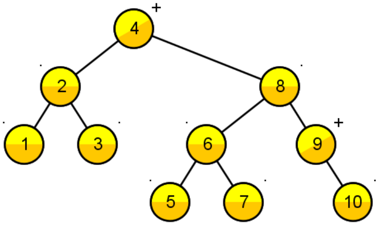



Figure: Inserting $1, \ldots, 10$ into an AVL-tree [Gna]

**Summary:**

- Historical the first search tree providing guaranteed `insert`, `remove` and `lookup` in $O(\log n)$
- However not amortized update costs of $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are (a,b)-trees

**(a,b)-Tree:**

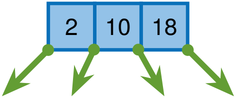
- Also known as **b-tree** (b for “balanced”)
- Used in databases and file systems

**Idea:**

- Save a varying number of elements per node
- So we have space for elements on an `insert` and balance operation

**(a,b)-Tree:**

- All leaves have the same depth
- Each inner node has $\geq a$ and $\leq b$ nodes
  (Only the root node may have less nodes)




- Each node with $n$ children is called "node of degree $n$" and holds $n-1$ sorted elements
- Subtrees are located "between" the elements
- We require: $a \geq 2$ and $b \geq 2a-1$

**(2,4)-Tree:**

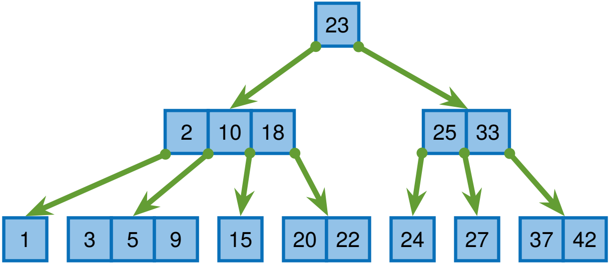



Figure: Example of an (2,4)-tree

- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

**Not an (2,4)-Tree:**

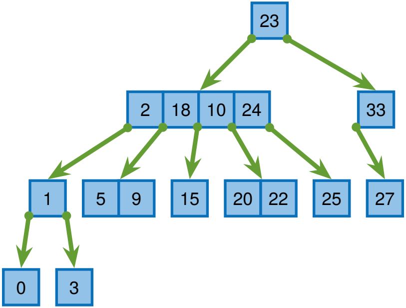



Figure: **Not** an (2,4)-tree

- Invalid sorting
- Degree of node too large / too small
- Leaves on different levels

**Searching an element:** (`lookup`)

- The same algorithm as in BinarySearchTree
- Searching from the root downwards
- The keys at each node set the path

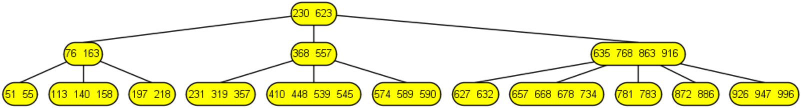



Figure: (3,5)-Tree [Gna]

**Inserting an element:** (`insert`)

- Search the position to insert the key into
- This position will always be an leaf
- Insert the element into the tree
- **Attention:** As a result node can overflow by one element (Degree $b+1$)
- Then we **split** the node

**Inserting an element:** (`insert`)

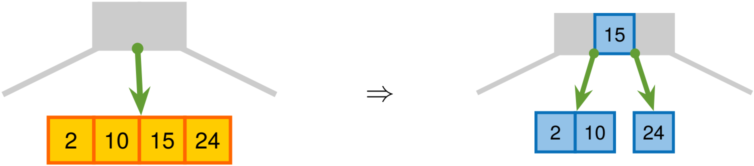



Figure: Splitting a node

- If the degree is higher than $b+1$ we split the node
- This results in a node with $\text{ceil}\left(\frac{b-1}{2}\right)$ elements, a node with $\text{floor}\left(\frac{b-1}{2}\right)$ elements and one element for the parent node
- Thats why we have the limit $b \geq 2a-1$

UNI FREIBURG

**Inserting an element:** (`insert`)

- If the degree is higher than $b+1$ we split the node
- Now the parent node can be of a higher degree than $b+1$
- We split the parent nodes the same way
- If we split the root node we create a new parent root node (The tree is now one level deeper)

**Removing an element:** (`remove`)

- Search the element in $O(\log n)$ time
- **Case 1:** The element is contained by a leaf
  - Remove element
- **Case 2:** The element is contained by an inner node
  - Search the successor in the right subtree
  - The successor is always contained by a leaf
  - Replace the element with its successor and delete the successor from the leaf
- **Attention:** The leaf might be too small (degree of $a - 1$)
  $\Rightarrow$ We rebalance the tree

**Removing an element:** (`remove`)

- **Attention:** The leaf might be too small (degree of $a-1$)
  $\Rightarrow$ We rebalance the tree
  - **Case a:** If the left or right neighbour node has a degree greater than a we **borrow** one element from this node

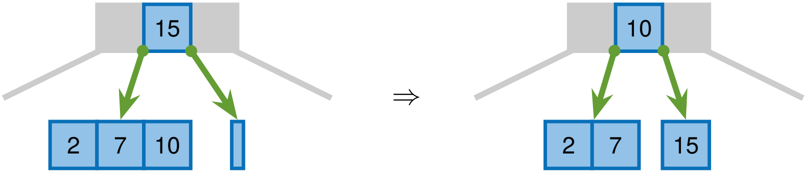



Figure: Borrow an element

**Removing an element:** (`remove`)

- **Attention:** The leaf might be too small (degree of $a - 1$)
  $\Rightarrow$ We rebalance the tree
  - **Case b:** We merge the node with its right or left neighbour

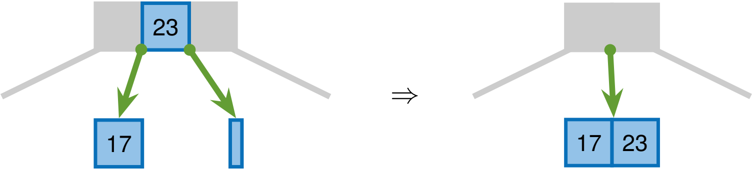



Figure: Merge two nodes

**Removing an element:** (`remove`)

- Now the parent node can be of degree $a - 1$
- We merge parent nodes the same way
- If the root has only a single child
  - Remove the root
  - Define sole child as new root
  - The tree shrinks by one level

**Runtime complexity of `lookup`, `insert` and `remove`:**

- All operations in $O(d)$ with $d$ being the depth of the tree
- Each node (except the root) has more than $a$ children
  $\Rightarrow n \geq a^{d-1}$ and $d \leq 1 + \log_a n = O(log_a n)$

**In detail:**

- `lookup` always takes $\Theta(d)$
- `insert` and `remove` often require only $O(1)$ time
- **Worst case:** split or merge all nodes on path up to the root
- Therefore instead of $b \geq 2a - 1$ we need $b \geq 2a$

UNI FREIBURG

**Counter example (2,3)-Tree:**

- Before executing `delete(11)`

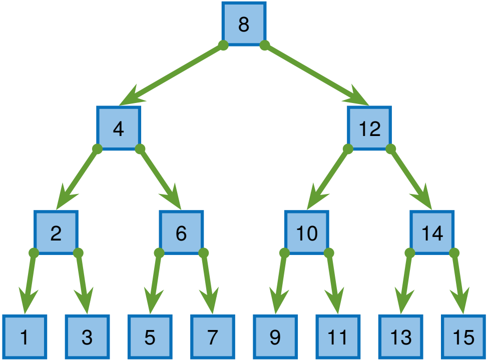



Figure: Normal (2,3)-Tree

**Counter example (2,3)-Tree:**

- Executing `delete(11)`

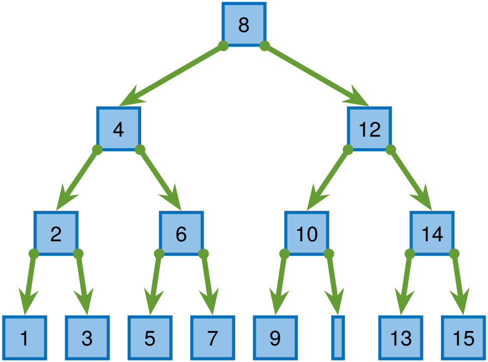



Figure: (2,3)-Tree - Delete step 1

**Counter example (2,3)-Tree:**

- Executing `delete(11)`

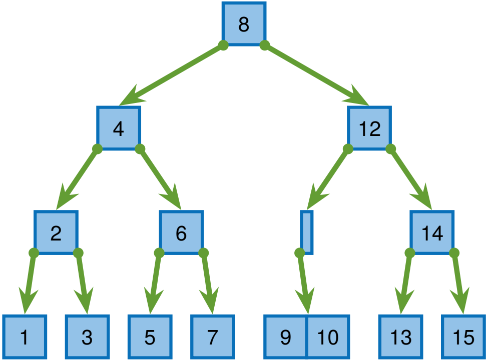



Figure: (2,3)-Tree - Delete step 2

**Counter example (2,3)-Tree:**

- Executing `delete(11)`

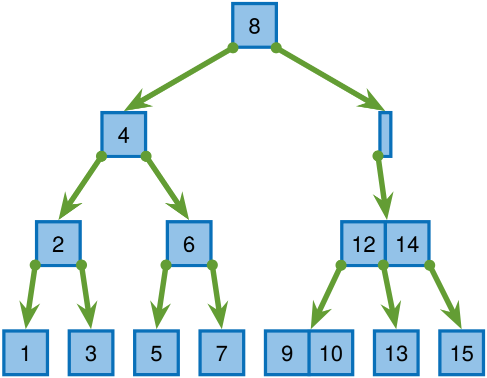



Figure: (2,3)-Tree - Delete step 3

## Counter example (2,3)-Tree:

- Executed `delete(11)`

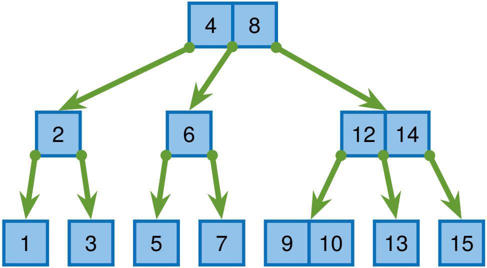



Figure: (2,3)-Tree - Delete step 4

## Counter example (2,3)-Tree:

- Executing `insert(11)`

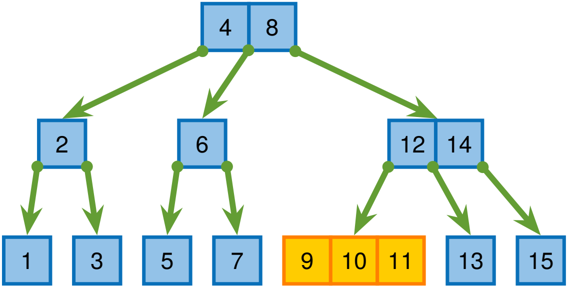



Figure: (2,3)-Tree - Insert step 1

**Counter example (2,3)-Tree:**

- Executing `insert(11)`

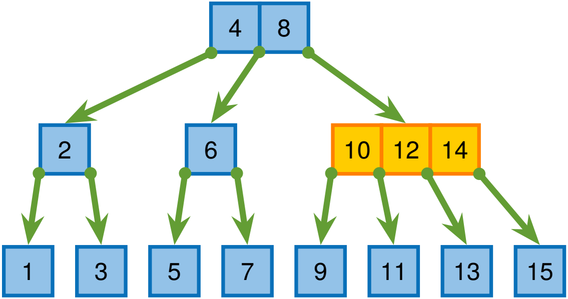



Figure: (2,3)-Tree - Insert step 2

**Counter example (2,3)-Tree:**

- Executing `insert(11)`

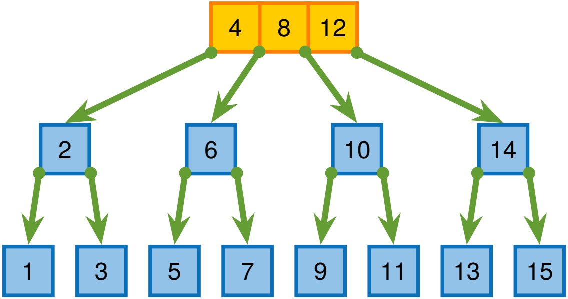



Figure: (2,3)-Tree - Insert step 3

**Counter example (2,3)-Tree:**

- Executed `insert(11)`

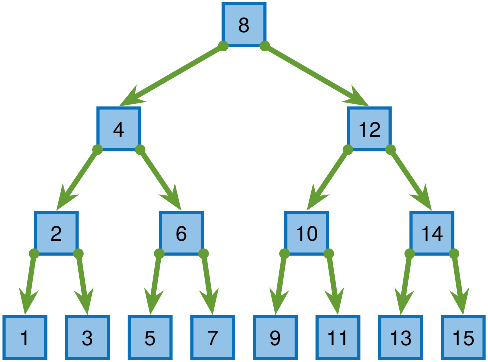



Figure: (2,3)-Tree - Insert step 4

**Counter example (2,3)-Tree:**

- We are exactly where we started
- If $b = 2a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$
- We need $b \geq 2a$ instead of $b \geq 2a - 1$

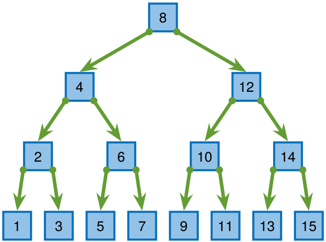



Figure: (2,3)-Tree

**(2,4)-Tree:**

- If all nodes have 2 children we have to merge the nodes up to the root on a `remove` operation
- If all nodes have 4 children we have to split the nodes up to the root on a `insert` operation
- If all nodes have 3 children it takes some time to reach one of the previous two states

⇒ **Nodes of degree 3 are stable**
Neither an `insert` nor a `remove` operation trigger rebalancing operations

UNI FREIBURG

**(2,4)-Tree:**

- **Idea:**
  - After an expensive operation the tree is in a stable state
  - It takes some time until the next expensive operation occurs
- Like with dynamic arrays:
  - Reallocation is expensive but it takes some time until the next expensive operation occurs
  - If we overallocate clever we have an amortized runtime of $O(1)$

**Terminology:**

- We analyze a sequence of $n$ operations
- Let $\Phi_i$ be the potential of the tree after the *i-th* operation
- $\Phi_i$ = the number of stable nodes with degree 3
- Empty tree has 0 nodes: $\Phi = 0$

**Example:**

- Nodes of degree 3 are highlighted

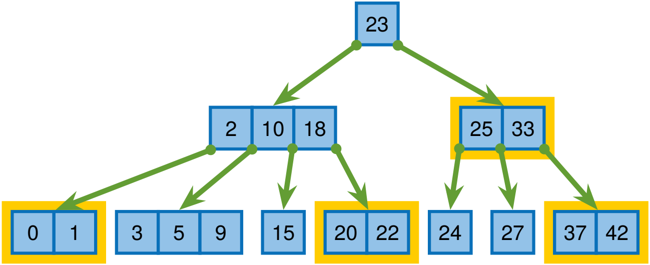



Figure: Tree with potential $\Phi = 4$

UNI FREIBURG

**Terminology:**

- Let $c_i$ be the costs = runtime of the $i$-th operation
- We will show:
  - Each operation can at most destroy one stable node
  - For each cost incurring step the operation creates an additional stable node
- The costs for operation $i$ are coupled to the difference of the potential levels

$$c_i \leq A \cdot \underbrace{(\Phi_i - \Phi_{i-1})} + B, \qquad A > 0 \text{ and } B > A$$

Number of gained stable nodes (degree 3) $\geq -1$

- Each operation has an amortitzed cost of $O(1)$ summing up to $O(n)$ in total

**Case 1:** *i-th* operation is an `insert` operation on a full node

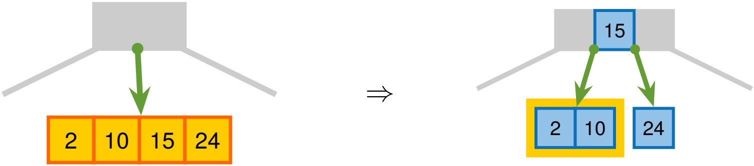



Figure: Splitting a node on `insert`

- Each splitted node creates a node of degree 3
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too

UNI FREIBURG

**Case 1:** *i-th* operation is an `insert` operation on a full node

- Let $m$ be the number of nodes split
- The potential rises by $m$
- If the "stop-node" is of degree 3 then the potential goes down by one

$$\Phi_i \geq \Phi_{i-1} + m - 1$$
$$\Rightarrow m \leq \Phi_i - \Phi_{i-1} + 1$$

Costs: $c_i \leq A \cdot m + B$

$$\Rightarrow c_i \leq A \cdot (\Phi_i - \Phi_{i-1} + 1) + B$$
$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + \underbrace{A + B}_{B'}$$

**Case 2:** *i-th* operation is an `remove` operation

- **Case 2.1:** Inner node
  - Searching the successor in a tree is $O(d) = O(\log n)$
  - Normally the tree is coupled with a doubly linked list
    $\Rightarrow$ We can find the succcessor in $O(1)$

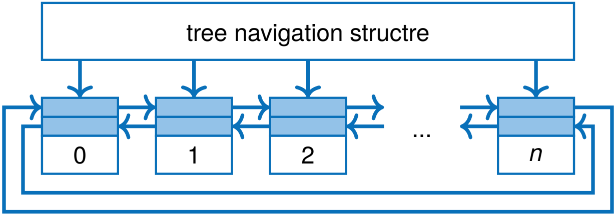



Figure: Tree with doubly linked list

**Case 2:** *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations
  - Case 2.1.1: Potential rises by one

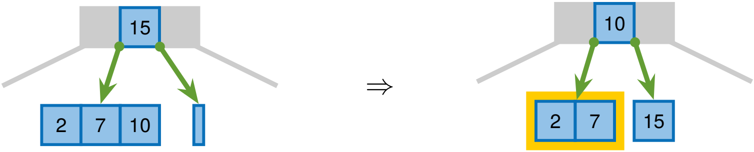



Figure: Case 2.1.1: Borrow an element

**Case 2:** *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations
  - Case 2.1.2: Potential is lowered by one

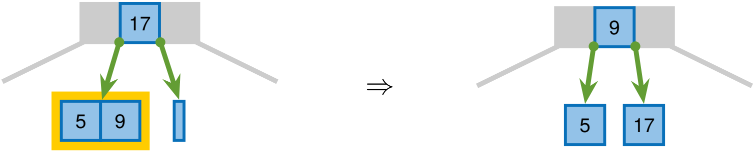



Figure: Case 2.1.2: Borrow an element

**Case 2:** *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

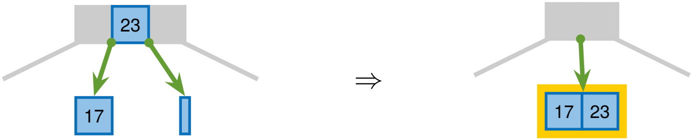



Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree $> 2$ or a node of degree 2, which can borrow from a neighbour

**Case 2:** *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

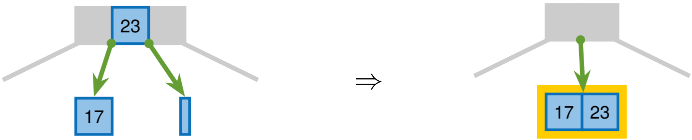



Figure: Merging two nodes

- The potential rises by $m$
- If the "stop-node" is of degree 2 then the potential eventually goes down by one
- Same costs as `insert`

UNI FREIBURG

**Lemma:**

- We know:

$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \qquad A > 0 \text{ and } B > A$$

- With that we can conclude:

$$\sum_{i=0}^{n} c_i \in O(n)$$

**Proof:**

$$\sum_{i=0}^{n} c_i \leq \underbrace{A \cdot (\Phi_1 - \Phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\Phi_2 - \Phi_1) + B}_{\leq c_2} + \cdots + \underbrace{A \cdot (\Phi_n - \Phi_{n-1}) + B}_{\leq c_n}$$

$$= A \cdot (\Phi_n - \Phi_0) + B \cdot n \qquad | \text{ telescope sum}$$

$$= A \cdot \Phi_n + B \cdot n \qquad | \text{ we start with an empty tree}$$

$$< A \cdot n + B \cdot n \in O(n) \qquad | \text{ number of degree 3 nodes}$$

$$< \text{number of nodes}$$

**Red-Black Tree:**

- Binary tree with red and black nodes
- Number of black nodes on path to leaves is equal
- Can be interpreted as (2,4)-tree (also named 2-3-4-tree)
- Each (2,4)-tree-node is a small red-black-tree with a black root node

# Red-Black-Trees

## Introduction

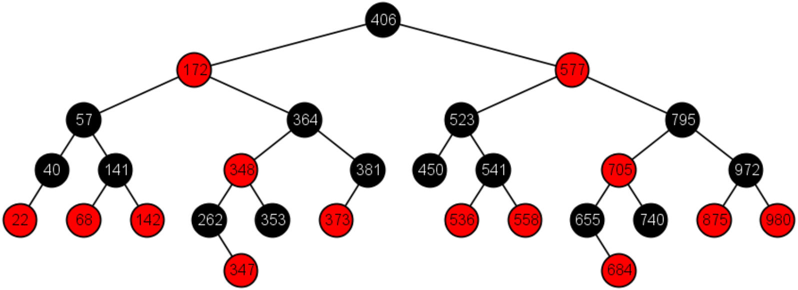



Figure: Example of an red-black-tree [Gna]

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
**Introduction to Algorithms.**
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
`https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf`.

- **Gnarley Trees**

[Gna] Gnarley Trees
`https://people.ksp.sk/~kuko/gnarley-trees/`

- **AVL-Tree**

[Wik] AVL tree
https://en.wikipedia.org/wiki/AVL_tree

- **(a,b)-Tree**

[Wika] 2-3-4 tree
https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934_tree

[Wikb] (a,b)-tree
https://en.wikipedia.org/wiki/(a,b)-tree

- **Red-Black-Tree**

[Wik] Red-black tree
https://en.wikipedia.org/wiki/Red%E2%80%93black_tree