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Algorithm Theory
Exercise Sheet 10

Due: Tuesday, 11th of January, 2022, 4 pm

Exercise 1: Randomized Dominating Set *(20 Points)*

Let $G = (V, E)$ be an undirected graph. A set $D \subseteq V$ is called a *dominating set* if each node in V is either contained in D or adjacent to a node in D. The problem is to find a dominating set which is as small as possible (note that $D = V$ is trivially a dominating set). However, the problem of finding a minimum dominating set (or even a constant approximation) is NP-hard. In this exercise we present a randomized algorithm for d-regular graphs (i.e., graphs in which each node has exactly d neighbors) that computes a $O(\log n)$-approximation of a minimum dominating set.

Let $c > 0$.

Algorithm 1 `domset`(G)

1: $D = \emptyset$
2: Each node joins D independently with probability $p := \min\{1, \frac{(c+2)\ln n}{d+1}\}$
3: Each node that is neither in D nor has a neighbor in D joins D
4: **return** D

For simplicity, in all tasks you may assume that $\frac{(c+2)\cdot\ln n}{d+1} \leq 1$, i.e., that $p = \frac{(c+2)\cdot\ln n}{d+1}$.

(a) Explain the runtime of `domset`. *(1 Point)*

(b) Show that `domset` returns a dominating set with an expected size of $O\left(\frac{n \log n}{d}\right)$.

Hint: Use the inequality $(1 - x) \leq e^{-x}$. *(4 Points)*

(c) Show that after line 2 of `domset`, D has size $O\left(\frac{n \log n}{d}\right)$ with probability at least $1 - \frac{1}{n^{c+1}}$.

Hint: For $v \in V$*, let* X_v *be the random variable with* $X_v = 1$ *if* v *joins* D *in line 2 and* $X_v = 0$ *else. Now use Chernoff's Bound.* *(3 Points)*

(d) Show that with probability at least $1 - \frac{1}{n^{c+1}}$, no node joins D in line 3 of `domset`. *(3 Points)*

(e) Conclude that `domset` returns a dominating set of size $O\left(\frac{n \log n}{d}\right)$ with probability at least $1 - \frac{1}{n^c}$. *(1 Point)*

(f) Someone might now say: "Why not doing parts (c)-(e) like this: Let X_v be the random variable with $X_v = 1$ if v is in D (at the end of the algorithm) and $X_v = 0$ else. Then use Chernoff's Bound."

What would you respond?

Hint: Read the slide from the lecture about Chernoff Bounds carefully. *(1 Point)*

(g) Finally, show that `domset` computes an $\mathcal{O}(\log n)$-approximation of a minimum dominating set (i.e., $D \in \mathcal{O}(|D^*| \log n)$ where D^* is a minimum dominating set) with probability at least $1 - \frac{1}{n^c}$. *(3 Points)*

We now have shown that `domset` is a Monte Carlo algorithm for the problem "$O(\log n)$ minimum dominating set approximation". That is, `domset` has a fixed deterministic runtime and a probabilistic correctness guarantee.

(h) Describe a Las Vegas algorithm for "$O(\log n)$ minimum dominating set approximation". That is, your algorithm must always return the correct answer and its runtime must be polynomial in expectation *and* w.h.p. Prove that your algorithm has these properties. *(4 Points)*