Exercise 1: Randomized Dominating Set

Let $G = (V, E)$ be an undirected graph. A set $D \subseteq V$ is called a dominating set if each node in $V$ is either contained in $D$ or adjacent to a node in $D$. The problem is to find a dominating set which is as small as possible (note that $D = V$ is trivially a dominating set). However, the problem of finding a minimum dominating set (or even a constant approximation) is NP-hard. In this exercise we present a randomized algorithm for $d$-regular graphs (i.e., graphs in which each node has exactly $d$ neighbors) that computes a $O(\log n)$-approximation of a minimum dominating set.

Let $c > 0$.

\begin{algorithm}
\textbf{domset}(G)
\begin{algorithmic}[1]
\State $D = \emptyset$
\State Each node joins $D$ independently with probability $p := \min\{1, \frac{(c+2)\ln n}{d+1}\}$
\State Each node that is neither in $D$ nor has a neighbor in $D$ joins $D$
\State \textbf{return} $D$
\end{algorithmic}
\end{algorithm}

For simplicity, in all tasks you may assume that $\frac{(c+2)\ln n}{d+1} \leq 1$, i.e., that $p = \frac{(c+2)\ln n}{d+1}$.

(a) Explain the runtime of \textbf{domset}. \hfill (1 Point)

(b) Show that \textbf{domset} returns a dominating set with an expected size of $O\left(\frac{n\log n}{d}\right)$. \hfill (4 Points)

\textbf{Hint}: Use the inequality $(1 - x) \leq e^{-x}$.

(c) Show that after line 2 of \textbf{domset}, $D$ has size $O\left(\frac{n\log n}{d}\right)$ with probability at least $1 - \frac{1}{n^{c+1}}$. \hfill (3 Points)

\textbf{Hint}: For $v \in V$, let $X_v$ be the random variable with $X_v = 1$ if $v$ joins $D$ in line 2 and $X_v = 0$ else. Now use Chernoff’s Bound.

(d) Show that with probability at least $1 - \frac{1}{n^{c+1}}$, no node joins $D$ in line 3 of \textbf{domset}. \hfill (3 Points)

(e) Conclude that \textbf{domset} returns a dominating set of size $O\left(\frac{n\log n}{d}\right)$ with probability at least $1 - \frac{1}{n^c}$. \hfill (1 Point)

(f) Someone might now say: “Why not doing parts (c)-(e) like this: Let $X_v$ be the random variable with $X_v = 1$ if $v$ is in $D$ (at the end of the algorithm) and $X_v = 0$ else. Then use Chernoff’s Bound.”

What would you respond?

\textbf{Hint}: Read the slide from the lecture about Chernoff Bounds carefully. \hfill (1 Point)

(g) Finally, show that \textbf{domset} computes an $O(\log n)$-approximation of a minimum dominating set (i.e., $D \in O(|D^*| \log n)$ where $D^*$ is a minimum dominating set) with probability at least $1 - \frac{1}{n^c}$. \hfill (3 Points)
We now have shown that \texttt{domset} is a Monte Carlo algorithm for the problem \textquotedblright\textit{O}(\log n) minimum dominating set approximation\textquotedblright. That is, \texttt{domset} has a fixed deterministic runtime and a probabilistic correctness guarantee.

(h) Describe a Las Vegas algorithm for \textquotedblright\textit{O}(\log n) minimum dominating set approximation\textquotedblright. That is, your algorithm must always return the correct answer and its runtime must be polynomial in expectation \textit{and} w.h.p. Prove that your algorithm has these properties. \hfill (4 Points)