Exercise 1: Randomized Coloring

Let $G = (V, E)$ be a simple, undirected graph with maximum degree $\Delta$. A vertex coloring of a graph is an assignment of colors to the vertices such that adjacent vertices have different colors. More formally, a coloring $\phi$ is a mapping $\phi : V \to C$ from $V$ to a color space $C$ such that $\phi(u) \neq \phi(v)$ if $\{u, v\} \in E$.

Consider the following randomized algorithm to compute a coloring of $G$ with $2\Delta$ colors, i.e., a coloring $\phi : V \to \{1, \ldots, 2\Delta\}$.

Each uncolored node $v$ assigns itself a tentative color $c_v \in \{1, \ldots, 2\Delta\}$ uniformly at random. If $v$ has no neighbor with the same (tentative or permanent) color, it keeps $c_v$ permanently. Otherwise it uncolors itself again. Repeat until all nodes are colored. In pseudocode:

```
Algorithm 1 color(G)
1: for v ∈ V do
2:   φ(v) = ⊥ > each node is initially uncolored
3: while there is a v with φ(v) = ⊥ do
4:   for each u with φ(u) = ⊥ independently do
5:     choose $c_u \in \{1, \ldots, 2\Delta\}$ uniformly at random
6:   for each u with φ(u) = ⊥ do
7:     if u has no neighbor w with $c_u = c_w$ or $c_u = φ(w)$ then
8:       φ(u) := $c_u$
```

We call one run of the while-loop in line 3 a round.

(a) Show that for each round and each uncolored node $u$, the probability that the condition in line 7 is true (i.e., $u$ permanently chooses a color) is at least $1/2$. (7 Points)

(b) Show that in each round, in expectation, the number of uncolored nodes is at least halved. (4 Points)

```
Hint: Use part (a).
```

(c) Show that `color` terminates in $O(\log n)$ rounds with high probability. That is, for a given $c > 0$, `color` terminates in $O(\log n)$ rounds with probability at least $1 - \frac{1}{n^c}$. (9 Points)

```
Hint: Use part (a).
```