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Algorithm Theory
Exercise Sheet 11

Due: Tuesday, 18th of January, 2022, 4 pm

Exercise 1: Randomized Coloring *(20 Points)*

Let $G = (V, E)$ be a simple, undirected graph with maximum degree Δ. A vertex coloring of a graph is an assignment of colors to the vertices such that adjacent vertices have different colors. More formally, a coloring ϕ is a mapping $\phi : V \to C$ from V to a color space C such that $\phi(u) \neq \phi(v)$ if $\{u, v\} \in E$.

Consider the following randomized algorithm to compute a coloring of G with 2Δ colors, i.e., a coloring $\phi : V \to \{1, \ldots, 2\Delta\}$.
Each uncolored node v assigns itself a tentative color $c_v \in \{1, \ldots, 2\Delta\}$ uniformly at random. If v has no neighbor with the same (tentative or permanent) color, it keeps c_v permanently. Otherwise it uncolors itself again. Repeat until all nodes are colored. In pseudocode:

Algorithm 1 `color`(G)

```
1: for v ∈ V do
2:     φ(v) = ⊥                                        ▷ each node is initially uncolored
3: while there is a v with φ(v) = ⊥ do
4:     for each u with φ(u) = ⊥ independently do
5:         choose c_u ∈ {1, ..., 2Δ} uniformly at random
6:     for each u with φ(u) = ⊥ do
7:         if u has no neighbor w with c_u = c_w or c_u = φ(w) then
8:             φ(u) := c_u
```

We call one run of the while-loop in line 3 a *round*.

(a) Show that for each round and each uncolored node u, the probability that the condition in line 7 is true (i.e., u permanently chooses a color) is at least $1/2$. *(7 Points)*

(b) Show that in each round, in expectation, the number of uncolored nodes is at least halved. *(4 Points)*

Hint: Use part (a).

(c) Show that `color` terminates in $O(\log n)$ rounds with high probability. That is, for a given $c > 0$, `color` terminates in $O(\log n)$ rounds with probability at least $1 - \frac{1}{n^c}$. *(9 Points)*

Hint: Use part (a).