Algorithm Theory

Exercise Sheet 13 - Bonus*

Due: Tuesday, 1st of February, 2021, 4 pm

*Bonus points can be earned normally, but do not increase the threshold for the “Studienleistung”.

Exercise 1: Online Vertex Cover  
(10* Points)

Let $G = (V, E)$ be an unweighted undirected graph. Consider the following online version of the minimum vertex cover problem. Initially, we are given the set of nodes $V$ and an empty vertex cover $S = \emptyset$. Then, the edges appear one-by-one in an online fashion. When a new edge $\{u, v\}$ appears, the algorithm needs to guarantee that the edge is covered (i.e., if this is not already the case, at least one of the two nodes $u$ and $v$ needs to be added to $S$). Once a node is in $S$ it cannot be removed from $S$.

(a) Provide a deterministic online algorithm with strict competitive ratio at most 2. (1 Point)

(b) Show that any deterministic online algorithm for the online vertex cover problem has strict competitive ratio at least 2. (4 Points)

(c) Use Yao’s principle to show that any randomized online algorithm for the online vertex cover problem has a strict competitive ratio at least $3/2$. (5 Points)

Exercise 2: Maximum Cut  
(10* Points)

Let $G = (V, E)$ be an unweighted undirected graph. A maximum cut of $G$ is a cut whose size is at least the size of any other cut in $G$. We consider an online version of the maximum cut problem, where the nodes $V$ of a graph $G = (V, E)$ appear in an online fashion. The algorithm should partition the nodes $V$ into two sets $A$ and $B$ such that induced cut is as large as possible. Whenever a new node appears we also learn its edges to the nodes that have already appeared before. An incoming node has to be immediately assigned to either $A$ or $B$ and that decision is final.

(a) Describe a deterministic online maximum cut algorithm with strict competitive ratio at least $1/2$. 
Hint: An online algorithm for a maximization problem is said to have strict competitive ratio $\alpha$ if it guarantees that $\text{ALG} \geq \alpha \cdot \text{OPT}$, where $\text{ALG}$ and $\text{OPT}$ are the solutions of the online algorithm and of an optimal offline algorithm, respectively. (5 Points)

(b) Show that no deterministic online algorithm for the online maximum cut problem can have a (constant) strict competitive ratio that is better (larger) than $1/2$. (5 Points)