Algorithm Theory

Chapter 7
Randomized Algorithms

Part VII:
Fast Randomized Minimum Cut Algorithm

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A Better Randomized Minimum Cut Alg.?

We saw: If the contraction algorithm is repeated $O(n^2 \log n)$ times, one can find a minimum cut in time $O(n^4 \log n)$, w.h.p.

- Time $O(n^4 \log n)$ is not very spectacular, a simple max flow based implementation has time $O(n^4)$.

However, we will see that the contraction algorithm is nevertheless very interesting because:

1. The algorithm can be improved to beat every known deterministic algorithm.

2. It allows to obtain strong statements about the distribution of cuts in graphs.

Better Randomized Algorithm

Recall:

- Consider a fixed min cut (A, B), assume (A, B) has size k
- The algorithm outputs (A, B) iff none of the k edges crossing (A, B) gets contracted.
- Throughout the algorithm, the edge connectivity is at least k and therefore each node has degree $\geq k$
- Before contraction i, there are $n + 1 - i$ nodes and thus at least $(n + 1 - i)k/2$ edges
- If no edge crossing (A, B) is contracted before, the probability to contract an edge crossing (A, B) in step i is at most

$$\frac{k}{\frac{(n+1-i)k}{2}} = \frac{2}{n+1-i}.$$

Improving the Contraction Algorithm

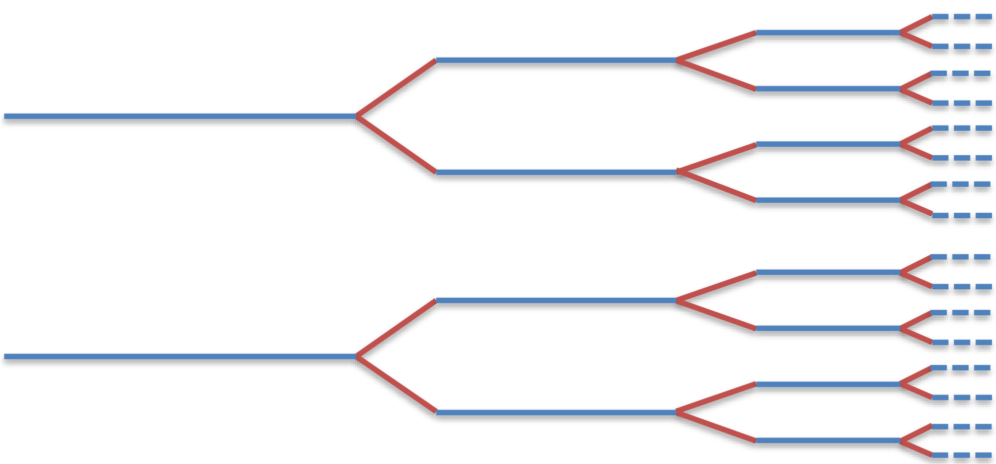
- For a specific min cut (A, B), if (A, B) survives the first $i - 1$ contractions,

$$\mathbb{P}(\text{edge crossing } (A,B) \text{ in contraction } i) \leq \frac{2}{n-i+1}.$$

Observation: The probability only gets large for large i

- **Idea:** The early steps are much safer than the late steps.

  Maybe we can repeat the late steps more often than the early ones.



Safe Contraction Phase

Lemma: A given min cut (A,B) of an n-node graph G survives the first $n - \left\lceil {}^{n}/_{\sqrt{2}} + 1 \right\rceil$ contractions, with probability $> {}^{1}/_{2}$.

Proof:

- Event $\mathcal{E}_i$: cut (A,B) survives contraction i
- Probability that (A,B) survives the first $n-t$ contractions:

$$\mathbb{P}\left(\bigcap_{i=1}^{n-t} \mathcal{E}_i\right) = \prod_{i=1}^{n-t} \mathbb{P}(\mathcal{E}_i | \mathcal{E}_1 \cap \dots \cap \mathcal{E}_{i-1}) \geq \prod_{i=1}^{n-t}\left(1 - \frac{2}{n-i+1}\right) = \prod_{i=1}^{n-t} \frac{n-i-1}{n-i+1}$$

$$= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \dots \cdot \frac{t+1}{t+3} \cdot \frac{t}{t+2} \cdot \frac{t-1}{t+1} = \frac{t(t-1)}{n(n-1)}$$

$$t = \left\lceil \frac{n}{\sqrt{2}} + 1 \right\rceil > \frac{n}{\sqrt{2}} + 1 \quad \Rightarrow \quad \mathbb{P}\left(\bigcap_{i=1}^{n-t} \mathcal{E}_i\right) > \frac{\left(\frac{n}{\sqrt{2}} + 1\right) \cdot \frac{n}{\sqrt{2}}}{n \cdot (n-1)} > \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

Better Randomized Algorithm

Let's simplify a bit:

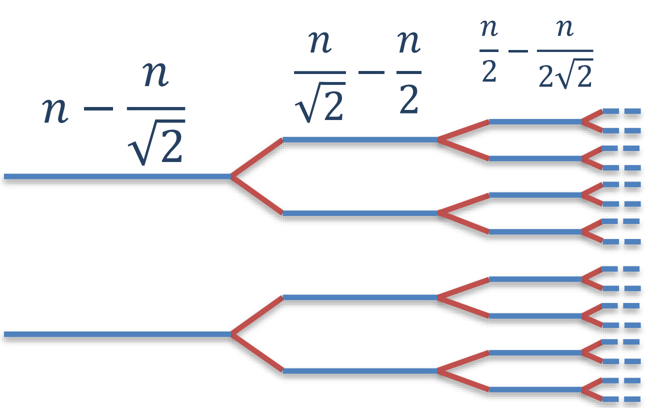
- Pretend that $n/\sqrt{2}$ is an integer (for all n we will need it).
- Assume that a given min cut survives the first $n - {}^{n}/_{\sqrt{2}}$ contractions with probability $\geq {}^{1}/_{2}$.

$\mathbf{contract}(\boldsymbol{G}, \boldsymbol{t})$:

- Starting with n-node graph G, perform $n - t$ edge contractions such that the new graph has t nodes.

$\mathbf{mincut}(\boldsymbol{G})$:

1. $X_1 \coloneqq \mathrm{mincut}\left(\mathrm{contract}\left(G, n/\sqrt{2}\right)\right);$
2. $X_2 \coloneqq \mathrm{mincut}\left(\mathrm{contract}\left(G, n/\sqrt{2}\right)\right);$
3. **return** $\min\{X_1, X_2\};$



Success Probability

Preserves min cut with prob. $\geq 1/2$

mincut(G):

1. $X_1 := \text{mincut}\left(\text{contract}\left(G, n/\sqrt{2}\right)\right);$
2. $X_2 := \text{mincut}\left(\text{contract}\left(G, n/\sqrt{2}\right)\right);$
3. **return** $\min\{X_1, X_2\};$

$\boldsymbol{P(n)}$: probability that the above algorithm returns a min cut when applied to a graph with n nodes.

- Probability that X_1 is a min cut $\geq \frac{1}{2} \cdot P\left(\frac{n}{\sqrt{2}}\right)$

Recursion:

$$P(n) \geq 1 - \left(1 - \frac{1}{2} \cdot P\left(\frac{n}{\sqrt{2}}\right)\right)^2 = P\left(\frac{n}{\sqrt{2}}\right) - \frac{1}{4} \cdot P\left(\frac{n}{\sqrt{2}}\right)^2, \qquad P(2) = 1$$

Success Probability

Theorem: The recursive randomized min cut algorithm returns a minimum cut with probability at least $1/\log_2 n$.

$$P(n) \geq \frac{1}{\log_2 n}$$

Proof (by induction on n):

$$P(n) = P\left(\frac{n}{\sqrt{2}}\right) - \frac{1}{4} \cdot P\left(\frac{n}{\sqrt{2}}\right)^2, \qquad P(2) = 1$$

Base case: $(n = 2)$ $P(2) \geq \frac{1}{\log_2 2} = 1$

induction hypothesis

Ind. step: $(n > 2)$ $P(n) \geq P\left(\frac{n}{\sqrt{2}}\right) - \frac{1}{4} \cdot P\left(\frac{n}{\sqrt{2}}\right)^2 \geq \frac{1}{\log_2\left({}^{n}/_{\sqrt{2}}\right)} - \frac{1}{4 \cdot \log_2\left({}^{n}/_{\sqrt{2}}\right)^2}$

$$= \frac{1}{\log_2 n - \frac{1}{2}} \cdot \left(1 - \frac{1}{4\log_2 n - 2}\right) = \frac{4\log_2 n - 3}{4\log_2^2 n - \underbrace{(4\log_2 n - 1)}_{> 3\log_2 n}}$$

$$> \frac{4\log_2 n - 3}{4\log_2^2 n - 3\log_2 n} = \frac{1}{\log_2 n}$$

Running Time

1. $X_1 := \text{mincut}\left(\text{contract}(G, n/\sqrt{2})\right);$
2. $X_2 := \text{mincut}\left(\text{contract}(G, n/\sqrt{2})\right);$
3. **return** $\min\{X_1, X_2\};$

Recurrence Relation:

- $T(n)$: time to apply algorithm to n-node graphs
- Recursive calls: $2T\left({}^{n}/_{\sqrt{2}}\right)$
- Number of contractions to get to ${}^{n}/_{\sqrt{2}}$ nodes: $O(n)$

$$T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2), \qquad T(2) = O(1)$$

Running Time

Theorem: The running time of the recursive, randomized min cut algorithm is $O(n^2 \log n)$.

Proof:

- Can be shown in the usual way, by induction on n
  - Or by applying the Master theorem…

Remarks:

- The running time is only by an $O(\log n)$-factor slower than the basic contraction algorithm.
- The success probability is exponentially better!
- For finding a minimum cut with high probability, we now need $O(\log^2 n)$ repetitions and we therefore obtain an overall running time of $O(n^2 \log^3 n)$.