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Algorithm Theory
Sample Solution Exercise Sheet 8

Due: Tuesday, 14th of December, 2021, 4 pm

Exercise 1: Work Schedule *(10 Points)*

Assume you want to design a work schedule for a hospital for the next n days. The hospital employs k doctors. On day i, exactly p_i doctors need to be present, for $i = 1, \ldots, n$. Each doctor j provides a set $L_j \subseteq \{1, \ldots, n\}$ of days on which he or she is willing to work.

(a) Describe a polynomial-time algorithm that either

- returns a list $L'_j \subseteq L_j$ of working days for each doctor j such that on day i, exactly p_i doctors are present or
- reports that there is no such set of lists that fulfills the given constraints.

(b) The hospital finds that the doctors tend to submit lists that are much too restrictive, and so it often happens that there is no feasible working schedule. Thus, the hospital relaxes the requirements in the following way. For some number $c > 0$, each doctor j can be forced to work on c days which are not in his/her list L_j.

Give a polynomial-time algorithm to solve this problem, i.e., the algorithm should either

- return a list L'_j of working days for each doctor j with $|L'_j \setminus L_j| \leq c$ such that on day i, exactly p_i doctors are present or
- report that there is no such set of lists that fulfills the given constraints.

Sample Solution

(a) We translate the problem to a flow network: There is a source node s and a sink t. For each doctor j, there is a node α_j and an edge from s to α_j with capacity $|L_j|$ (any capacity $\geq |L_j|$ would work). Then we introduce a new layer of nodes with one node β_i for each day $i \in \{1, \ldots, n\}$ and draw an edge with capacity 1 from α_j to β_i if $i \in L_j$. Finally we draw an edge with capacity p_i from β_i to t.

We compute a max flow with integer flow values. There is a feasible work schedule iff the maximum flow of the network equals $\sum_{i=1}^{n} p_i$. Doctor j has to work on day i if there is a flow (of value 1) from α_j to β_i.

(b) We extend the network from above: For each doctor j we introduce an additional node α'_j and an edge from s to α'_j with capacity c and an edge from α'_j to β_i with capacity 1 if $i \notin L_j$.

We compute a max flow with integer flow values. There is a feasible work schedule iff the maximum flow of the network equals $\sum_{i=1}^{n} p_i$. Doctor j has to work on day i if there is a flow (of value 1) from either α_j or α'_j to β_i.

Exercise 2: Minimum Cut with Maximum Edges *(10 Points)*

Let $G = (V, E, c)$ be a flow network with positive integer capacities and $s, t \in V$. Give an efficient algorithm to find a minimum s-t cut with the largest number of edges. That is, we want to find an s-t cut (A, B) which

- is minimum (w.r.t. the edge weights)
- among all minimum s-t cuts, has a maximum number of edges going from A to B.

Show that your algorithm is correct and analyze its runtime as a function of $|E|$ and C (minimum s-t cut capacity of G).

Sample Solution

We define a new flow network $G' = (V, E, c')$ from $G = (V, E, c)$ by setting $c'(e) = (m + 1) \cdot c(e) - 1$. For a cut (A, B) it follows that $c'(A, B) = (m + 1)c(A, B) - k$ where $k = \#(\text{cut edges}) \leq m$. Hence we have $c(A, B) = \left\lceil \frac{c'(A,B)}{(m+1)} \right\rceil$.

Let (A, B) and (X, Y) be two cuts with $c'(A, B) \leq c'(X, Y)$. It follows that

$$c(A, B) = \left\lceil \frac{c'(A, B)}{(m + 1)} \right\rceil \leq \left\lceil \frac{c'(X, Y)}{(m + 1)} \right\rceil = c(X, Y)$$

Hence a minimum cut in G' is also a minimum cut in G. Now assume that (A, B) and (X, Y) are two minimum cuts in G and (A, B) has more edges. We obtain that $c'(A, B) < c'(X, Y)$. Hence a minimum cut in G' is a minimum cut with the largest number of edges in G.

We find a minimum cut in G' by first computing a maximum flow and then finding a minimum cut in $O(m)$ as described in the lecture. If C is the maximum flow value in G, then G' has a maximum flow of $O(mC)$, so runtime is $O(m^2C)$.