University of Freiburg
Dept. of Computer Science
Prof. Dr. F. Kuhn
P. Bamberger, P. Schneider

# Algorithm Theory
# Sample Solution Exercise Sheet 9

**Due:** Tuesday, 21st of December, 2021, 4 pm

## Exercise 1: Hypergraph Matching *(10 Points)*

A *hypergraph* is a generalization of an undirected graph in which edges consist of arbitrary subsets of vertices. That is, a hypergraph consists of a set of nodes $V$ and a set of edges $E \subseteq \mathcal{P}(V)$. A hypergraph is called $k$-uniform if each edge has size $k$, i.e., contains exactly $k$ nodes. For example, a simple, undirected graph is a 2-uniform hypergraph.
A matching of a hypergraph is a set of edges which are pairwise disjoint. The definitions of a maximal and maximum matching are extended to hypergraphs in the obvious way.

(i) Show that a maximum matching of a $k$-uniform hypergraph has size at most $\lfloor \frac{n}{k} \rfloor$.

(ii) For an arbitrary $k \geq 2$, provide a $k$-uniform hypergraph $G$ and a maximal matching $M$ such that $|M| = \frac{|M^*|}{k}$ where $M^*$ is a maximum matching.

(iii) Show that for any maximal matching $M$ of a $k$-uniform hypergraph we have $|M| \geq \frac{|M^*|}{k}$ where $M^*$ is a maximum matching.

## Sample Solution

(i) If a matching contains $\ell$ edges, the graph contains at least $\ell \cdot k$ nodes. Hence, the number of edges can not be larger than $\frac{n}{k}$.

(ii) $G = (V, E)$
$V = \{v_{ij} \mid 1 \leq i, j \leq k\}$
$E = \big\{\{v_{i1}, \dots, v_{ik}\} \mid 1 \leq i \leq k\big\} \cup \big\{\{v_{11}, \dots, v_{k1}\}\big\}$

$M = \big\{\{v_{11}, \dots, v_{k1}\}\big\}$
$M^* = \big\{\{v_{i1}, \dots, v_{ik}\} \mid 1 \leq i \leq k\big\}$

(iii) Let $e_1, \dots, e_\ell$ be the edges of $M$. For each $i = 1, \dots, \ell$ let $S_i := \{e \in E \mid e \cap e_i \neq \emptyset\}$. As $M$ is maximal, we have $S_1 \cup \ldots \cup S_\ell = E$. We also have $|S_i \cap M^*| \leq k$, because each edge in $S_i \cap M^*$ contains one node of $e_i$ and $|e_i| = k$. We therefore have

$$|M^*| = |E \cap M^*| = |(S_1 \cap M^*) \cup \ldots \cup (S_\ell \cap M^*)| \leq \ell \cdot k = k|M| \ .$$

## Exercise 2: Randomization *(10 Points)*

Assume you are given a randomized algorithm $\mathcal{A}$ that given a graph $G$ with $n$ nodes and a maximum matching of size $s$, computes an integer $k \leq s$ in time $T(n)$ such that with probability at least $p(n)$ we have $k = s$.
Give an algorithm with running time $\mathcal{O}\big(p(n)^{-1} \cdot T(n) \cdot \log n\big)$ that computes the size of a maximum matching of a graph with $n$ nodes with probability at least $1 - \frac{1}{n}$. Prove the success probability.

## Sample Solution

We repeat $\mathcal{A}$ for $p^{-1} \ln n$ times and take the maximum of all outputs. The probability that the size of a maximum matching is not among the outputs is at most

$$(1-p)^{p^{-1} \ln n} \leq e^{-\ln n} = \frac{1}{n}$$