Exercise 1: Randomized Coloring  

Let \( G = (V, E) \) be a simple, undirected graph with maximum degree \( \Delta \). A vertex coloring of a graph is an assignment of colors to the vertices such that adjacent vertices have different colors. More formally, a coloring \( \phi \) is a mapping \( \phi : V \rightarrow C \) from \( V \) to a color space \( C \) such that \( \phi(u) \neq \phi(v) \) if \( \{u, v\} \in E \).

Consider the following randomized algorithm to compute a coloring of \( G \) with \( 2\Delta \) colors, i.e., a coloring \( \phi : V \rightarrow \{1, \ldots, 2\Delta\} \).

Each uncolored node \( v \) assigns itself a tentative color \( c_v \in \{1, \ldots, 2\Delta\} \) uniformly at random. If \( v \) has no neighbor with the same (tentative or permanent) color, it keeps \( c_v \) permanently. Otherwise it uncolors itself again. Repeat until all nodes are colored. In pseudocode:

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Algorithm 1 color(G)
1: for v ∈ V do
2:  \phi(v) = ⊥ > each node is initially uncolored
3: while there is a v with \phi(v) = ⊥ do
4:   for each u with \phi(u) = ⊥ independently do
5:     choose \( c_u \in \{1, \ldots, 2\Delta\} \) uniformly at random
6:     for each u with \phi(u) = ⊥ do
7:       if u has no neighbor w with \( c_u = c_w \) or \( c_u = \phi(w) \) then
8:          \phi(u) := c_u
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We call one run of the while-loop in line 3 a round.

(a) Show that for each round and each uncolored node \( u \), the probability that the condition in line 7 is true (i.e., \( u \) permanently chooses a color) is at least 1/2.  

(b) Show that in each round, in expectation, the number of uncolored nodes is at least halved.  

*Hint: Use part (a).*

(c) Show that \texttt{color} terminates in \( O(\log n) \) rounds with high probability. That is, for a given \( c > 0 \), \texttt{color} terminates in \( O(\log n) \) rounds with probability at least \( 1 - \frac{1}{n^c} \).  

*Hint: Use part (a).*

Sample Solution

(a) Consider an uncolored node \( u \) and a neighbor \( w \). The probability that \( c_u = c_w \) or \( c_u = \phi(w) \) is \( \frac{1}{2\Delta} \). With a union bound it follows that the probability that \( u \) can not keep its color is at most \( \frac{1}{2\Delta} = \frac{1}{2} \).
(b) Let $U$ be the set of uncolored nodes at the beginning of the round. For each $u \in U$, let $X_u = 1$ if $u$ remains uncolored and $X_u = 0$ if $u$ gets colored. Then the expected number of nodes remaining uncolored is

$$E\left[\sum_{u \in U} X_u\right] = \sum_{u \in U} E[X_u] = \sum_{u \in U} \Pr(u \text{ remains uncolored}) \leq \frac{|U|}{2}$$

(c) The probability that $u$ is uncolored after $(c + 1) \log n$ rounds is at most $\frac{1}{2^{(c+1)\log n}} = \frac{1}{n^{c+1}}$. A union bound over all nodes yields that the probability that there is an uncolored node after $(c + 1) \log n$ rounds is at most $\frac{n}{n^{c+1}} = \frac{1}{n^c}$.

Alternative solution: W.l.o.g. we can assume that $c \geq 1$ (otherwise we can choose $c' = \max\{c, 1\}$ and obtain an even better bound). We call a round successful if at least half of the uncolored nodes keep their color. Note that at the latest after $\log n$ successful rounds, all nodes are permanently colored. Let $X_i$ be the random variable with $X_i = 1$ if round $i$ is successful and $X_i = 0$ otherwise.

From (a) follows that $\Pr(X_i = 1) \geq 1/2$. Let $X = \sum_{i=1}^{16c\log n} X_i$. We have $\mu := E[X] \geq 8c\log n$. Chernoff’s Bound yields

$$\Pr(X \leq \log n) \leq \Pr(X \leq 4c\log n) \leq \Pr(X \leq (1 - 1/2)\mu) < e^{-\mu/8} \leq e^{-c\log n} = \frac{1}{n^c}.$$ 

So with high probability, there are at least $\log n$ successful rounds among the first $16c\log n = O(\log n)$ rounds.