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Algorithm Theory
Sample Solution Exercise Sheet 11

Due: Tuesday, 18th of January, 2022, 4 pm

Exercise 1: Randomized Coloring *(20 Points)*

Let $G = (V, E)$ be a simple, undirected graph with maximum degree Δ. A vertex coloring of a graph is an assignment of colors to the vertices such that adjacent vertices have different colors. More formally, a coloring ϕ is a mapping $\phi : V \to C$ from V to a color space C such that $\phi(u) \neq \phi(v)$ if $\{u, v\} \in E$.

Consider the following randomized algorithm to compute a coloring of G with 2Δ colors, i.e., a coloring $\phi : V \to \{1, \ldots, 2\Delta\}$.
Each uncolored node v assigns itself a tentative color $c_v \in \{1, \ldots, 2\Delta\}$ uniformly at random. If v has no neighbor with the same (tentative or permanent) color, it keeps c_v permanently. Otherwise it uncolors itself again. Repeat until all nodes are colored. In pseudocode:

Algorithm 1 `color`(G)

```
1: for v ∈ V do
2:     φ(v) = ⊥                                          ▷ each node is initially uncolored
3: while there is a v with φ(v) = ⊥ do
4:     for each u with φ(u) = ⊥ independently do
5:         choose c_u ∈ {1, ..., 2Δ} uniformly at random
6:     for each u with φ(u) = ⊥ do
7:         if u has no neighbor w with c_u = c_w or c_u = φ(w) then
8:             φ(u) := c_u
```

We call one run of the while-loop in line 3 a *round*.

(a) Show that for each round and each uncolored node u, the probability that the condition in line 7 is true (i.e., u permanently chooses a color) is at least $1/2$. *(7 Points)*

(b) Show that in each round, in expectation, the number of uncolored nodes is at least halved. *(4 Points)*

Hint: Use part (a).

(c) Show that `color` terminates in $O(\log n)$ rounds with high probability. That is, for a given $c > 0$, `color` terminates in $O(\log n)$ rounds with probability at least $1 - \frac{1}{n^c}$. *(9 Points)*

Hint: Use part (a).

Sample Solution

(a) Consider an uncolored node u and a neighbor w. The probability that $c_u = c_w$ or $c_u = \phi(w)$ is $\frac{1}{2\Delta}$. With a union bound it follows that the probability that u can not keep its color is at most $\frac{\Delta}{2\Delta} = \frac{1}{2}$.

(b) Let U be the set of uncolored nodes at the beginning of the round. For each $u \in U$, let $X_u = 1$ if u remains uncolored and $X_u = 0$ if u gets colored. Then the expected number of nodes remaining uncolored is

$$E[\sum_{u \in U} X_u] = \sum_{u \in U} E[X_u] = \sum_{u \in U} \Pr(u \text{ remains uncolored}) \leq \frac{|U|}{2}$$

(c) The probability that u is uncolored after $(c+1)\log n$ rounds is at most $\frac{1}{2^{(c+1)\log n}} = \frac{1}{n^{c+1}}$. A union bound over all nodes yields that the probability that there is an uncolored node after $(c+1)\log n$ rounds is at most $\frac{n}{n^{c+1}} = \frac{1}{n^c}$.

Alternative solution: W.l.o.g. we can assume that $c \geq 1$ (otherwise we can choose $c' = \max\{c, 1\}$ and obtain an even better bound). We call a round successful if at least half of the uncolored nodes keep their color. Note that at the latest after $\log n$ successful rounds, all nodes are permanently colored. Let X_i be the random variable with $X_i = 1$ if round i is successful and $X_i = 0$ otherwise. From (a) follows that $\Pr(X_i = 1) \geq 1/2$. Let $X = \sum_{i=1}^{16c\log n} X_i$. We have $\mu := E[X] \geq 8c\log n$. Chernoff's Bound yields

$$\Pr(X \leq \log n) \leq \Pr(X \leq 4c\log n) \leq \Pr(X \leq (1-1/2)\mu) < e^{-\mu/8} \leq e^{-c\log n} = \frac{1}{n^c} .$$

So with high probability, there are at least $\log n$ successful rounds among the first $16c\log n = O(\log n)$ rounds.